# Portfolio Credit Risk:
# A Model of Correlated Credit Losses Dynamics and the Inverse-Gamma Approximation


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This version: August 2008


### Abstract

'Top-down' approach of portfolio credit risk lacks of transparency in terms of explicit implications at the 'down' level. 'Bottom-up' approach is less flexible to accommodate for modeling the dynamics of credit loss. In this paper, a hybrid approach is proposed, which combines both the modeling parsimony of 'top-down' models and the explicit implications of 'bottom-up' models in terms of default risk of single names. Correlated forward credit losses of the $n$ constituents of the pool (or index) are modeled under a HJM-like framework. This allows for arbitrage-free dynamics of single names' defaults matching the $n$ single names' CDS spread curves and an explicit characterization of a name-sensitive credit loss correlation structure. Under an explicit setup of correlated lognormal cumulated credit losses, it is argued that the Inverse-Gamma distribution (with time-varying shape and scaling parameters generated back at the 'down' level by the HJM-like model) approximates the credit pool loss distribution, which results in a simple closed-form solution of CDO spreads. Calibration is made efficiently thanks to a name grouping technique and shows a high fitting power of the model, particularly for the skew of the market CDO spreads. Calibration procedure, model extensions and numerical examples illustrating the impact of the subprime crisis are discussed. Furthermore, Monte Carlo methods for the simulation of the Inverse-Gamma pool loss dynamics are developed to price CDO derivatives with early-exercise-style and/or loss-trigger features.



**JEL classifications:** G13.

**Keywords:** Portfolio credit loss; CDO; Credit correlation



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# 1 Introduction

**Literature and Motivations**

The popularity of multi-name credit derivatives has led to numerous innovations in terms of the modeling of credit loss and dependence of default events (see Burtschell, Gregory and Laurent (2005) and Andersen and Sidenius (2005) for literature surveys). While the pioneer approach based on the copula technique has been adopted as a market standard for vanilla CDO products, the pricing of more sophisticated credit portfolio derivatives requires an explicit modeling of the portfolio loss dynamics. Although some recent papers enhance the insertion of copulas to the dynamic loss setup, such as the common Poisson shock model of Brigo, Pallavicini and Torresetti (2007), it is a matter of fact that the 'bottom-up' approach (consisting of modeling single names' defaults before moving to the aggregate level of the pool loss) is less flexible to handle the issue of portfolio loss dynamics.

In this regard, the aggregate forward-loss approach known as the 'top-down' approach, initiated by Bennani (2005), Andersen, Piterbarg and Sidenius (2005), Schönbucher (2005) and Giesecke and Goldberg (2005), has been seen as a promising alternative to copula models. The idea consists of leaving the modeling of single names' defaults, and focus on the explicit modeling of the aggregate pool loss. By following this avenue, one only needs to model the dynamics of the pool loss rather than single names' defaults and their correlation. This recent literature exhibits a rich diversity in terms of modeling features, which besides the pioneer JHM-like models, such as Bennani (2005) and Schönbucher (2005), includes the periodic impulses model of Hull and White (2007) as well as Markov chain models such as the Spread-Loss Model of Arnsdorf and Halperin (2007).[1]

While 'top-down' models show in general a high capacity to fit the correlation skew observed over the market and accommodate for an easier pricing of CDO derivatives, their massive use is presently limited by their lack of transparency in terms of implications at the 'down' level of single names. Indeed, by modeling directly the aggregate pool loss, these models leave us without any answer neither on the dynamics of single names' defaults nor on the credit correlation that are consistent with the inferred aggregate loss dynamics. Of course, one can recover an implied base credit correlation from the model after calibration to market data of CDO tranches, but this correlation structure is not informative in itself, since it just obtained by inverting the Gaussian copula to the fitted model. Further, because of this opacity, 'top-down' models fail to predict how single names' defaults and their correlation structure will evolve over time, which remains the same intrinsic limit of copula models. This is crucial point when dealing with CDO derivatives, since the opacity of 'top-down' models in terms of implied dynamics of single names' defaults and credit correlation becomes a serious issue whenever one targets a tight control over the model risk (either for trading strategies or hedging purposes).

**Proposed Model**

In this paper, we propose an alternative avenue for the pricing of multi-name products that could be qualified as half way between the 'bottom-up' and the 'top-down' approaches. The main idea

[1] This is not an exhaustive listing of this recent and fast growing literature on the explicit modeling of portfolio loss dynamics.

consists of making use of the higher flexibility offered by the 'top-down' models of forward aggregate loss by applying it at the 'down' level to model the forward loss of single names, which allows for arbitrage-free dynamics of single names' defaults and explicit characterization of the correlation structure between them (i.e., no more opacity). As one would expect, to make this approach feasible, we need an aggregation tool permitting us to move form the single names universe to the aggregate pool level (which could be viewed as the equivalent to the copula technique in 'bottom-up' models); and this is what precisely constitutes the main contribution of the paper.

More precisely, we start by modeling the forward loss at the single names level under a Gaussian HJM-like framework, where the dynamics of instantaneous loss rates are matched to the single names' curves of CDS spreads. This allows us to build an arbitrage-free model of lognormal cumulated credit loss (LCCL) for single names, where credit loss correlation is explicitly captured by both a time-deterministic instantaneous correlation of the single names' credit loss rates and a name-specific HJM-Markovian volatility function (involving the drift matched to the term structure of single names' CDS spreads). Once this LCCL model of single names is defined, we move to the aggregation of individual losses. To do this, we are inspired from well-known techniques of distribution transformation used in the actuarial literature on aggregate claims in insurance risk and previous established results in continuous martingales used in the pricing of Asian options to argue the use of the inverse-Gamma distribution as a pricing tool of the aggregate pool loss as resulting form our LCCL model of single names' losses. Interestingly, the inverse-Gamma model admits a closed-form solution for CDO spreads, very easy to implement, that ensures the mapping of the parameters of the LCCL model of correlated credit loss rates of single names to the aggregate pool loss distribution. We spend a large part of the paper to describe the calibration procedure, make it *user-friendly* by reducing the high-dimension problem arising from the LCCL model of single names, and study the explicit implications of the model at the 'down' level.

While preparing this paper, we learned that the model of Jäckel (2008) is built in the same spirit than our model in the sense it is at a half-way between the 'bottom-up' and 'top-down' approaches and proposes a similar characterization of the aggregate pool loss distribution. However, the two models are different to the extent how single names' defaults are modeled and pooled as well as how the aggregate pool loss distribution is first inferred and then used for CDO derivatives valuation purposes.

**Results and Applications**
As most of 'top-down' models, our model maintains the attractive feature to be calibrated, for different maturities, to the entire correlation skew formed by the CDO tranches spreads. But it adds to this the capacity of matching the term structure of CDS spreads of all the index constituents (i.e., the single names composing the CDS index). To make the calibration of the model to the single names efficient, we employ a name grouping technique.

Two main points arise from the calibration of the model to the market data of CDO tranches. i) The model highly performs for fitting market spreads before the subprime credit crisis. The fitting errors of CDO spreads are found quite negligible, which justify the relevance of our setup of correlated single names' forward losses. ii) The fitting quality of the model, however, is less attractive when market quotations corresponding to the current environment of subprime crisis are of interest. To enhance this last outcome, we generalize the model further, which considerably improves its fitting power under such extreme market context. The extension consists of considering an aggregate loss distribution given by a mixture of the inverse-Gamma distribution and an *extra fat-tail leg* introduced to appropriately capture the market-implied likelihood of extreme losses.

We spend the end part of the paper explaining how the model could be used to price CDO derivatives. We also illustrate carefully different procedures leading to efficient numerical implementation of the model using Monte Carlo method in order to price early-exercise-style CDO derivatives.

# 2 A Credit Loss Rate Model for Single Names

## 2.1 *Prerequesties*

We provide here a simple model of single credit entities based on the concept of credit loss rate. We assume that uncertainty in financial markets is described by the probability space $(\Omega, \mathcal{F}, \mathbb{Q})$, where there is unique risk-neutral measure $\mathbb{Q}$ under which the time $t$-conditional expectation operator $\mathbb{E}_t^{\mathbb{Q}}[.] = \mathbb{E}^{\mathbb{Q}}[.|\mathcal{F}_t]$ is defined. The instantaneous short rate, we denote by $r$, is held constant in our analysis.

Let $P^M(0,T)$ denotes the time 0-market value of a corporate discount bond promising a face value of 1 dollar at time $T$. Define $\zeta$ as the random default time of the underlying credit entity. Without loss of generality, let the recovery value upon default equal to zero (i.e., a loss-given-default equalizing 100%). We are able to solve for the distribution $\mathbb{Q}[\zeta \leq s]$ such that we have,

$$P^M(0,T) = \mathbb{E}_0^{\mathbb{Q}}\left[e^{-rT}\mathbf{1}\{\zeta > T\}\right], \tag{2.1}$$

with $\mathbf{1}\{\mathcal{A}\}$ is the indicator function of event $\mathcal{A}$.

Let's define $(N_t)_{t\geq 0}$ as the process of *outstanding notional* as *implied* from $P^M(0,T)$, where the condition of solvency at current time 0 (i.e., $\zeta > 0$) is respected by adopting the convention $N_0 = 1\$$ (in general, $N_0$ must equal the nominal value). Then, we introduce *the conditional credit loss rate* $(x_t)_{t\geq 0}$ governing the evolution of $(N_t)_{t\geq 0}$,

$$dN_t = -x_t N_t dt, \qquad x_0 = 0, \tag{2.2}$$

which is equivalent to,

$$N_t = N_0 e^{-\int_0^t x_s ds}, \tag{2.3}$$

such that we satisfy the arbitrage-free condition,

$$P^M(0,T) = \frac{e^{-rT}}{N_0}\mathbb{E}_0^{\mathbb{Q}}[N_T] = e^{-rT}\mathbb{E}_0^{\mathbb{Q}}\left[e^{-\int_0^T x_s ds}\right]. \tag{2.4}$$

Equation (2.2) means that the outstanding notional will decrease in time due to occurred credit losses, where the time $t$-instantaneous credit loss $dN_t$ is determined by the conditional credit loss rate $x_t$ proportionally to the prior outstanding notional $N_t$.

Rather than modeling the dynamics of the default time $\zeta$, we are able to build an arbitrage-free process of *projected outstanding notional* matching market quotations and equivalent to the pair (Default probability, Loss-given-default) yielded by any structural default risk model matching the same market quotations. Of course, default events tend to occur suddenly over time, rather than continuously as implicitly implied by the credit loss rate model above. In addition, the loss-given-default is most likely suffered once default event occurs rather than progressively amortized over time as assumed by the credit loss rate model. But here, two main reasons would give motivation to adopt the credit loss rate (CLR) approach. First, once we are able to calibrate the CLR model to the term structure of market spreads, the two approaches (CLR model and structural model) are equivalent in terms of credit spreads. Second, as long as the concern is to price multi-name/correlation credit products such as CDO, the implicit assumptions behind the CLR model are consistent with the *way* market investors usually interpret and price asset portfolios. Indeed, a similar and older portfolio-risk problem that has been solved in the same way is the case of mortgage-backed-securities (MBS). Individual mortgage loans prepay when interest rates fall down significantly (prepayment risk affects severely the cash flows to be generated by a mortgage pool by transforming the known contractual schedule of cash flows into randomly distributed cash flows). Similarly to default events of credit entities, prepayment events observed over individual mortgage lines tend to occur suddenly. The method by which market investors price MBS consists of assuming that prepayment flows generated by a pool of mortgage lines are governed by a conditional prepayment rate, where the outstanding capital or notional of underlying individual mortgages is supposed to follow the same equation (2.2) above (after substituting the credit loss rate by some conditional prepayment rate and taking into account scheduled capital amortization). Once the stochastic function representing the conditional prepayment rate is calibrated (using for instance market spreads of MBS), prepayment flows and total cash flows generated by each individual mortgage line are projected based on this same prepayment rate function.

This is exactly the intuition behind our model. In this paper, we shall assume that credit losses suffered over the single names constituting the credit portfolio or index are governed by specific CLRs processes that have to be calibrated to market quotations. The main difference between our approach and which adopted by 'top-down' models is that we allow for single name-based CLRs and some correlation between them, rather than dealing with the aggregate credit loss at the portfolio level directly. This ensures that the calibrated CLRs processes are consistent at the same time with both CDO quotations and the CDS spreads of the pool constituents, with the main advantage of inferring explicitly an implied arbitrage-free structure of credit correlation.

The rest of the paper deals with these two key points: the arbitrage-free formulation of the CLR and the pricing of portfolio-wide credit risk. In the rest of this section, we provide a formal framework under which arbitrage-free CLRs are characterized. In Section 3, we show how single names' CLRs are pooled together to determine the portfolio aggregate loss. Calibration examples are provided in Section 4 and further extensions of the model are proposed in Section 5.

### 2.2 *A HJM-like Model of Credit Loss Rate*

#### 2.2.1 *The Forward Outstanding Notional*

Henceforth, the current time is set at zero and the time 0-outstanding notional is denoted by $N_0$. We start by defining the credit loss $L_t$ cumulated over the time interval $[0,t]$, we call cumulated credit loss (CCL), as

$$L_t = N_0 - N_t = N_0 \left[1 - e^{-\int_0^t x_s ds}\right]. \tag{2.5}$$

To price credit portfolio risk appropriately without implying arbitrage opportunities, the CCL process $(L_t)_{t\geq 0}$ must exhibit the following main properties: i) positive CLL: $L_t \geq 0$ for any $t > 0$ with $L_0 = 0$; ii) time-increasing CLL: $L_t \geq L_s$ for any $t \geq s$; iii) bounded CLL: $L_t \leq N_0$ for any $t > 0$; and iv) bounded CLL increment $dL_t$.

For any $0 \leq t < T$, let's define $N^f(t,T)$ as the time *T-forward outstanding notional* as,

$$N^f(t,T) := \mathbb{E}_t^{\mathbb{Q}}\left[N_T\right], \tag{2.6}$$

such that,

$$P^M(t,T) = e^{-r(T-t)} \frac{N^f(t,T)}{N_0}, \tag{2.7}$$

with the associated time *T-forward credit loss* $L^f(t,T)$ is given by the martinagle process,

$$L^f(t,T) := \mathbb{E}_t^{\mathbb{Q}}\left[L_T\right] = N_0 - N^f(t,T). \tag{2.8}$$

#### 2.2.2 *The Instantenous Credit Loss Rate*

Similarly to interest rate models, the arbitrage-free dynamics of the *spot CLR process* $(x_t)_{t\geq 0}$, and those of the CCL process $(L_t)_{t\geq 0}$, could be formulated using the HJM methodology.[2] By letting $(f(t,T))_{0\leq t\leq T}$ denote *the instantaneous forward CLR*, verifying:

$$\begin{aligned}
f(t,t) &:= x_t \quad \text{for any } t \geq 0, & (2.9)\\
f(t,s) &:= x_s \quad \text{for any } s \leq t, & (2.10)\\
N^f(t,T) &:= N_0 \exp\left(-\int_0^t x_s ds\right) \exp\left(-\int_t^T f(t,s)ds\right), & (2.11)
\end{aligned}$$

we have,

$$\begin{aligned}
N^f(t,T) &= N_0 \exp\left(-\int_0^T f(t,s)ds\right), & (2.12)\\
f(t,T) &= -\frac{\partial}{\partial T} \ln N^f(t,T), & (2.13)\\
P^M(t,T) &= e^{-r(T-t)} \exp\left(-\int_0^T f(t,s)ds\right). & (2.14)
\end{aligned}$$

[2] The following developments of the CLR dynamics under the HJM framework are similar to those proposed by Bennani (2005). Readers familiarized with the HJM-like formulation of forward credit loss could directly move to the next Subsection.

Notice here that the only divergence between this forward CLR model and standard forward interest rate models is that the outstanding notional at time $t \geq 0$, supporting the forward notional $N^f(t,T)$, is not re-initialized at $N_0$ in contrast to default-free bonds, but determined by both the CCL $L_t$ over the prior time interval $[0,t]$ and the projected CLR over the interval $[t,T]$.

Now, based on the concepts introduced earlier, we are able to introduce the dynamics of the forward CLR. The model we propose in this paper as well as the subsequent analysis are based on the class of one-factor HJM-like models. The extension to multi-factor models is easy to provide, but we shall demonstrate later in the paper that under our framework of correlated single names' CLRs, a one-factor model is able to calibrate well market quotations. Formally, we assume that the forward CLR dynamics are governed by a drift $\mu(.)$ and a bounded volatility function $\sigma(.)$ as follows:

$$df(t,T) = \mu(t,T)dt + \sigma(t,T)dW(t), \tag{2.15}$$

with $(W(t))_{t\geq 0}$ is a standard $\mathbb{Q}$-Brownian motion. Solving the SDE of $f(t,T)$ leads to,

$$\begin{aligned} f(t,T) &= f(0,T) + \int_0^t \mu(s,T)ds + \int_0^t \sigma(s,T)dW(s), && (2.16)\\ x_t &= f(t,t) = f(0,t) + \int_0^t \mu(s,t)ds + \int_0^t \sigma(s,t)dW(s). && (2.17) \end{aligned}$$

Let $F(t,T) := -\int_t^T f(t,s)ds$ be the running integral of the forward CLR, we have

$$-F(t,T) = \int_t^T f(0,s)ds + \int_t^T \int_0^t \mu(u,s)duds + \int_t^T \int_0^t \sigma(u,s)dW(u)ds. \tag{2.18}$$

Using the fact that

$$\int_t^T f(0,s)ds = -\left[F(0,T) + \int_0^t x_s ds\right], \tag{2.19}$$

and rearranging the iterated integrals involving the drift term and the diffusion component (by the mean of Fubini's theorem), we obtain

$$\begin{aligned} -F(t,T) &= -\left[F(0,T) + \int_0^t x_s ds\right] + \int_0^t \int_u^T \mu(u,s)dsdu + \int_0^t \int_u^T \sigma(u,s)dsdW(u), && (2.20)\\ dF(t,T) &= x_t - \int_t^T \mu(t,s)ds - \left(\int_t^T \sigma(t,s)ds\right) dW(t). && (2.21) \end{aligned}$$

Finally, it is easy to make use of the diffusion equation above of $F(t,T)$ and the fact that,

$$N^f(t,T) = N_0 \exp\left(-\int_0^t x_s ds\right) \exp\left(F(t,T)\right),$$

to deduce after a simple application of the Itô formula that the forward notional's diffusion equation obeys to,

$$\frac{dN^f(t,T)}{N^f(t,T)} = \left[-\int_t^T \mu(t,s)ds + \tfrac{1}{2}\left(\Sigma(t,T)\right)^2\right]dt - \Sigma(t,T)dW(t), \tag{2.22}$$

$$\Sigma(t,T) \equiv \int_t^T \sigma(t,u)du. \tag{2.23}$$

As a consequence, the standard HJM's arbitrage-free condition according to which the forward notional $N^f(t,T)$ is a martingale process (imposing a zero-drift condition to the SDE above) leads to the usual HJM's drift function under the CLR model:

$$\mu(t,T) = \sigma(t,T)\Sigma(t,T). \tag{2.24}$$

It then follows that the arbitrage-free spot CLR is given by,

$$x_t = f(0,t) + \int_0^t \sigma(s,t)\Sigma(s,t)ds + \int_0^t \sigma(s,t)dW(s). \tag{2.25}$$

## 2.3 A *Lognormal Cumulated Credit Loss* (*LCCL*) *Model*

We focus now on a class of models of Markovian spot CLR $(x_t)_{t\geq 0}$ easy to implement and calibrate, under which the volatility structure is given by,

$$\sigma(t,T) = \xi(t)h(T), \tag{2.26}$$

where $h(.)$ and $\xi(.)$ are deterministic functions (see Carverhill (1994)). By making use of the explicit solution of $x_t$ derived earlier, we have

$$x_t = f(0,t) + J(t) + h(t)\int_0^t \xi(s)dW(s), \tag{2.27}$$

$$J(t) \equiv h(t)\int_0^t \xi^2(s)\int_s^t h(u)duds. \tag{2.28}$$

Differentiating yields the spot CLR's SDE,

$$dx_t = \left[\psi(t) + \varphi(t)x_t\right]dt + \sigma(t,t)dW(t), \tag{2.29}$$

$$\psi(t) = \frac{\partial f(0,t)}{\partial t} + J'(t) - \varphi(t)\left[f(0,t) + J(t)\right], \tag{2.30}$$

$$\varphi(t) = \frac{h'(t)}{h(t)}, \tag{2.31}$$

$$\sigma(t,t) = \xi(t)h(t). \tag{2.32}$$

This means that choosing $h'(t) \neq 0$ will introduce a mean-reversion pattern (more precisely $h'(t) < 0$). We will restrict our analysis to the class of Markovian models above with the volatility structure verifying,

$$\begin{aligned}
\sigma(t,T) &= \bar{\sigma} e^{-a(T-t)}, && (2.33)\\
\xi(t) &= \bar{\sigma} e^{at}, && (2.34)\\
h(T) &= e^{-aT}. && (2.35)
\end{aligned}$$

This is a standard Gaussian model of CLR *à la* Hull-White, where we have

$$\begin{aligned}
dx_t &= [\psi(t) - a x_t]\, dt + \bar{\sigma} dW(t), && (2.36)\\
\psi(t) &= \frac{\partial f(0,t)}{\partial t} + a f(0,t) + \frac{\bar{\sigma}^2}{2a}\left(1 - e^{-2at}\right). && (2.37)
\end{aligned}$$

with,

$$\begin{aligned}
X(0,T) \quad &: = -\int_0^T x_t dt = F(0,T) - \int_0^T J(t)dt - \int_0^T (T-t)\xi(t) dW(t), && (2.38)\\
F(0,T) &= -\int_0^T f(0,t)dt = \left[rT + \ln P^M(0,T)\right] = -T s^M(T), && (2.39)
\end{aligned}$$

where $s^M(T)$ is defined as the market credit spread in excess of the short rate $r$ for the term $T$ as obtained from the market CDS spreads. It is easy to show that the CLR function $X(0,T)$ is normally distributed with,

$$\begin{aligned}
X(0,T) &= \mathbb{E}_0^{\mathbb{Q}}\left[X(0,T)\right] - \bar{\sigma}\int_0^T B(t,T) dW(t), && (2.40)\\
\mathbb{E}_0^{\mathbb{Q}}\left[X(0,T)\right] &= -T s^M(T) - \tfrac{1}{2} v(0,T) && (2.41)\\
\mathbb{V}\mathrm{ar}_0^{\mathbb{Q}}\left[X(0,T)\right] &= v(0,T), && (2.42)
\end{aligned}$$

where

$$\begin{aligned}
B(t,T) &= \frac{1}{a}\left[1 - e^{-a(T-t)}\right], && (2.43)\\
v(t,T) &= \left(\frac{\bar{\sigma}}{a}\right)^2 \left[(T-t) - 2B(t,T) + \left(\frac{1 - e^{-2a(T-t)}}{2a}\right)\right]. && (2.44)
\end{aligned}$$

Under this model, the outstanding notional at any time $T$, equal to $N_0 e^{X(0,T)}$, is lognormally distributed, which implies that the CCL given by $N_0\left[1 - e^{X(0,T)}\right]$ will exhibit a displaced-lognormal distribution.

# 3 The Aggregate Pool Loss Distribution

Let's now consider a credit portfolio of $n$ credit entities indexed by $i = 1, 2...n$, and a notional allocation $(w_i)_{i=1,...n}$, with $w_i \in (0,1)$ and $\sum_{i=1,...,n} w_i = 1$. To simplify notations, $(N_t, L_t)$ will denote henceforth the outstanding notional and CCL of the credit portfolio, while $(N_{i,t}, L_{i,t})$ denote the same variables for the credit entity $i$. Knowing that for any $i = 1, 2...n$,

$$\begin{aligned} L_{i,t} &= N_{i,0} - N_{i,t} && (3.1)\\ &= N_{i,0}\left[1 - e^{X_i(0,t)}\right] && (3.2)\\ &= w_i N_0 \left[1 - e^{X_i(0,t)}\right], && (3.3) \end{aligned}$$

and taking the convention $w_i = \frac{1}{n}$ for any $i$, we have

$$L_t = \sum_{i=1}^{n} L_{i,t} = N_0 \left[1 - \frac{1}{n}\sum_{i=1}^{n} e^{X_i(0,t)}\right]. \quad (3.4)$$

Before moving on, it is useful to introduce here the *excess credit loss* defined as follows:

$$\begin{aligned} \mathbf{EL}(b,t) \quad &: = \mathbb{E}_0^{\mathbb{Q}}\left[\max\left(0, \ell_t - b\right)\right], \qquad b \in [0,1], && (3.5)\\ &= \mathbb{E}_0^{\mathbb{Q}}\left[\ell_t \mathbf{1}\left\{\ell_t \geq b\right\}\right] - b\mathbb{Q}\left[\ell_t \geq b\right], && (3.6) \end{aligned}$$

with $\ell_t$ is the normalized aggregate pool loss cumulated by time $t$ given by,

$$\ell_t := \left(\frac{L_t}{N_0}\right) \quad \text{for any } t > 0. \quad (3.7)$$

We are interested in the distribution of the variable $\ell_t$ upon which we will be able to price CDO tranches as well as CDO derivatives.

## 3.1 *Credit Correlation Structure*

Under the LCCL model introduced earlier, we have for each reference name $i$,

$$\begin{aligned} dx_{i,t} &= \left[\psi_i(t) - a_i x_{i,t}\right] dt + \bar{\sigma}_i dW_i(t), && (3.8)\\ \psi_i(t) &= \frac{\partial f_i(0,t)}{\partial t} + a_i f_i(0,t) + \frac{\bar{\sigma}_i^2}{2a_i}\left(1 - e^{-2a_i t}\right). && (3.9) \end{aligned}$$

where $L_{i,T} = N_{i,0}\left[1 - e^{X_i(0,T)}\right]$ exhibits a displaced-lognormal distribution with,

$$\begin{aligned} \bar{L}_i(T) \quad &: = \mathbb{E}_0^{\mathbb{Q}}\left[L_{i,T}\right] = N_{i,0}\left[1 - \exp\left(m_i(T) + \tfrac{1}{2}\upsilon_i(0,T)\right)\right] = \frac{1}{n} N_0 \left[1 - e^{-Ts_i^M(T)}\right], && (3.10)\\ \bar{L}_{ii}(T) \quad &: = \mathbb{E}_0^{\mathbb{Q}}\left[L_{i,T}^2\right] = \frac{1}{n^2} N_0^2 \exp\left(2\left(m_i(T) + \upsilon_i(0,T)\right)\right) && (3.11)\\ \sigma_{L_i}^2(T) \quad &: = \mathbb{V}\mathrm{ar}_0^{\mathbb{Q}}\left[L_{i,T}\right] = \bar{L}_{ii}(T) - \bar{L}_i^2(T), && (3.12) \end{aligned}$$

with

$$m_i(T) := \mathbb{E}_0^{\mathbb{Q}}\left[X_i(0,T)\right] = -Ts_i^M(T) - \tfrac{1}{2}\upsilon_i(0,T). \quad (3.13)$$

We provide now the credit correlation structure as implied from the correlation structure of CLRs. We start by observing that for any pair of CCLs, $(L_{i,T}, L_{j,T})$ with $T > 0$, corresponding to two credit entities $i$ and $j$, the CCL correlation is given by

$$\rho_{ij}^{L}(T) := \frac{\bar{L}_{ij}(T) - \bar{L}_i(T)\bar{L}_j(T)}{\sigma_{L_i}(T)\sigma_{L_j}(T)}, \tag{3.14}$$

with,

$$\begin{aligned} \bar{L}_{ij}(T) := \mathbb{E}_0^{\mathbb{Q}}\left[L_{i,T}L_{j,T}\right] = \frac{1}{n^2}N_0^2 \\ \times \left[1 + \mathbb{E}_0^{\mathbb{Q}}\left[e^{X_i(0,T)+X_j(0,T)}\right] - \mathbb{E}_0^{\mathbb{Q}}\left[e^{X_i(0,T)}\right] - \mathbb{E}_0^{\mathbb{Q}}\left[e^{X_j(0,T)}\right]\right] \end{aligned} \tag{3.15}$$

$$\begin{aligned} \mathbb{E}_0^{\mathbb{Q}}\left[e^{X_i(0,T)+X_j(0,T)}\right] &= \exp\left(\left(m_i(T) + m_j(T)\right)\right) \\ &\times \exp\left(\tfrac{1}{2}\left(\upsilon_i(0,T) + \upsilon_j(0,T) + 2\upsilon_{i,j}(0,T)\right)\right) \end{aligned} \tag{3.16}$$

$$\upsilon_{i,j}(0,T) \quad : \quad = \mathbb{C}\mathrm{ov}_0^{\mathbb{Q}}\left[X_i(0,T), X_j(0,T)\right] \tag{3.17}$$

$$= \quad \bar{\sigma}_i\bar{\sigma}_j \int_0^T B_i(t,T)B_j(t,T)\rho_{i,j}(t)dt \tag{3.18}$$

where $\rho_{ij}(t)$ denotes the instantaneous correlation of spot CLRs,

$$\rho_{i,j}(t)dt = dW_i(t)dW_j(t). \tag{3.19}$$

We assume a time-increasing instantaneous correlation across the $n$ reference names taking the form,

$$\rho_{i,j}(t) = 1 - e^{-\gamma t} \text{ for any } (i,j),$$

with $\gamma > 0$ is a constant parameter capturing the shape of the instantaneous correlation curve. This yields the following expression of the covariance term $\upsilon_{i,j}(.)$ involved in the credit correlation function $\rho_{ij}^{L}(T)$,

$$\begin{aligned} \upsilon_{i,j}(t,T) = \left(\frac{\bar{\sigma}_i\bar{\sigma}_j}{a_ia_j}\right)\Bigg\{\left[(T-t) - B_i(t,T) - B_j(t,T) + \left(\frac{1-e^{-(a_i+a_j)(T-t)}}{a_i+a_j}\right)\right] \\ - \left[\left(\frac{e^{-\gamma t} - e^{-\gamma T}}{\gamma}\right) - H_i(t,T) - H_j(t,T) + H_{i,j}(t,T)\right]\Bigg\}, \end{aligned} \tag{3.20}$$

with

$$H_i(t,T) \quad = \quad \frac{1}{a_i - \gamma}\left[e^{-\gamma T} - e^{-\gamma t - a_i(T-t)}\right], \tag{3.21}$$

$$H_{i,j}(t,T) \quad = \quad \frac{1}{a_i + a_j - \gamma}\left[e^{-\gamma T} - e^{-\gamma t-(a_i+a_j)(T-t)}\right]. \tag{3.22}$$

Interestingly, although the instantaneous correlation $\rho_{i,j}(t)$ of CLRs is unchanged for any pair $(i,j)$, the level of credit loss correlation $\rho_{ij}^{L}(t)$ depends on the CDS spreads and the CLRs processes of the involved reference names, which makes this correlation intensity specific to the credit entities considered. This provides a rich structure of default correlation that is not allowed by previous models.

### 3.2 *Aggregate Pool Loss and the Inverse-Gamma Distribution Model*

It is well known that the sum of correlated lognormals does not behave as a lognormal by itself. The approximation of the distribution of the sum of correlated lognormals was of interest in two different fields. The first is related to the pricing of Asian options (for instance, see Geman and Yor (1993) and Milevsky and Posner (1998)), where the arithmetic average of the lognormal underlying price plays the same role of the variable $Z_t$ in our CDO pricing problem. The second is little far from financial market applications, and is with respect to the distribution of aggregate insurance claims studied in the actuarial literature (for instance, see Daykin, Pentikäinen and Pesonen (1994), Chapter 4). In the insurance field, however, the sum of correlated lognormals is not the only the case considered. Rather, the approximation methods deal with the general problem of random sums of correlated variables, where the marginal lognormal distribution is one among different cases of interest. For this reason, the methods we find are not specific to the marginal distribution of individual claims and usually employ the first three or four moments of the random sum (representing the aggregate claims in that case) to approximate the random sum distribution by the mean of transformation techniques. The transformation techniques generally share the same idea that consists of the symmetrization of the random sum distribution by solving for a parameter making the skewness of the transformed sum collapsing to zero, which makes possible the use of the standard normal distribution or the Gamma distribution depending on the method employed. Some methods permit also to control for the fat-tail by taking into account the kurtosis of the random sum into the transformation function.

The most known transformation techniques are the *Haldane transformation techniques* and the *Wilson-Hilferty' Gamma transformation technique* (see Pentikäinen (1987) for a detailed review). Although these transformation techniques are not employed here, they provide us with useful insights. Knowing that there is some equivalence between the Wilson-Hilferty' Gamma transformation function and the Haldane transformation function (see Pentikäinen (1987) for more details), both these two methods could be viewed as Gamma distribution-based transformation techniques (after some restriction introduced on the most general Haldane's Normal polynomial-based technique). This is in line with the formal result of Milevsky (1999) for continuous martingales, used by Milevsky and Posner (1998) to solve the valuation problem of (arithmetic) Asian options, and according to which the infinite sum of correlated lognormals (the continuous-time arithmetic average of the underlying in the context of Asian options) converges in probability to the Inverse-Gamma (IG) function.[3] The only required condition is that the Gaussian variables involved in the correlated lognormals exhibit negative means, which is equivalent to a negative drift condition imposed to the underlying asset process. Milevsky and Posner use the IG distribution of the infinite sum as approximation of the probability density function (p.d.f.) of the finite sum in the real case of discontinuous-time average. Although the conditions ensuring the convergence to the infinite

[3] As we will show below, there is a fundamental reciprocal relationship between the Gamma and the IG distributions.

sum limit distribution are *a priori* unsatisfied in real-cases, the aforementioned authors report that numerical results show a very good accuracy provided by the IG fit. The same good fit provided by the IG function was also reported by Pentikäinen (1987) in the context of aggregate insurance claims.

Based on the established results discussed above, and given the fact that $m_i(T) < 0$ for any single name $i$, we shall approximate the distribution of the normalized aggregate pool loss $\ell_t$ by the IG function. More precisely, fixing the time date $t$ (we will see later how incorporate the term structure dimension into our pool loss distribution problem), we assume that the p.d.f. of $\ell_t$ is given by a truncated IG function:

$$\mathbb{Q}\left[\ell_t \in d\ell\right] = \hat{g}_I\left(\ell | \alpha_t, \beta_t; c\right) d\ell, \qquad 0 < c \leq 1, \qquad \alpha_t, \beta_t > 0, \tag{3.23}$$

where $\hat{g}_I(.)$ is the IG p.d.f. $g_I(.)$ truncated by the upper bound of $c$, i.e.,

$$\hat{g}_I\left(\ell | \alpha_t, \beta_t; c\right) := \frac{g_I\left(\ell | \alpha_t, \beta_t\right)}{G_I\left(c | \alpha_t, \beta_t\right)}, \tag{3.24}$$

$$G_I\left(c | \alpha_t, \beta_t\right) := \int_0^c g_I\left(y | \alpha_t, \beta_t\right) dy, \tag{3.25}$$

with the truncation parameter $c$ is introduced to make the IG distribution satisfying the arbitrage-free condition,

$$\ell_t \in [0, 1] \quad \text{for any } t > 0. \tag{3.26}$$

It is useful to notice that the IG's p.d.f. $g_I(.)$ and cumulative distribution function (c.d.f.) $G_I(.)$ could be expressed in terms of the Gamma's p.d.f. $g(.)$ and c.d.f. $G(.)$ as follows:

$$g_I\left(y | \alpha, \beta\right) = \frac{1}{y^2} g\left(\frac{1}{y} \middle| \alpha, \beta\right), \tag{3.27}$$

$$G_I\left(y | \alpha, \beta\right) = 1 - G\left(\frac{1}{y} \middle| \alpha, \beta\right), \tag{3.28}$$

with

$$g\left(y | \alpha, \beta\right) = \frac{\beta^{-\alpha}}{\Gamma(\alpha)} y^{\alpha-1} \exp\left(-\frac{y}{\beta}\right), \qquad y \in (0, \infty), \tag{3.29}$$

and $\Gamma(.)$ is the Euler-Gamma function. Now, using the relationship above between $g_I(.)$ and $g(.)$ as well as the recurrence property of the Gamma p.d.f.,

$$g\left(y | \alpha, \beta\right) = \left(\frac{y^p}{\beta^p(\alpha-1)...(\alpha-p)}\right) g\left(y | \alpha - p, \beta\right), \qquad \text{for } p = 1, 2...,$$

it can be easily shown under the assumption of the IG distribution that the following closed-form solutions hold provided $\alpha_t > p$ :

$$\mathbb{E}_0^{\mathbb{Q}}\left[\ell_t^p\right] = \frac{1}{\beta_t^p(\alpha_t-1)(\alpha_t-2)...(\alpha_t-p)} \left[\frac{1 - G\left(\frac{1}{c} \middle| \alpha_t - p, \beta_t\right)}{1 - G\left(\frac{1}{c} \middle| \alpha_t, \beta_t\right)}\right], \qquad p = 1, 2..., \tag{3.30}$$

and for any constant $b \in [0,1]$,

$$\mathbb{Q}[\ell_t \leq b] = \left[\frac{1 - G\left(\frac{1}{b}\middle|\,\alpha_t, \beta_t\right)}{1 - G\left(\frac{1}{c}\middle|\,\alpha_t, \beta_t\right)}\right], \tag{3.31}$$

$$\mathbb{E}_0^{\mathbb{Q}}[\ell_t \mathbf{1}\{\ell_t \leq b\}] = \frac{1}{\beta_t(\alpha_t - 1)}\left[\frac{1 - G\left(\frac{1}{b}\middle|\,\alpha_t - 1, \beta_t\right)}{1 - G\left(\frac{1}{c}\middle|\,\alpha_t, \beta_t\right)}\right], \qquad \alpha_t > 1. \tag{3.32}$$

Interestingly, the two last equations directly lead to a closed-form solution of the excess credit loss under the IG distribution:

$$\begin{aligned}\mathbf{EL}(b,t) = \left(\frac{1}{\beta_t(\alpha_t - 1)}\right)&\left[\frac{G\left(\frac{1}{b}\middle|\,\alpha_t - 1, \beta_t\right) - G\left(\frac{1}{c}\middle|\,\alpha_t - 1, \beta_t\right)}{1 - G\left(\frac{1}{c}\middle|\,\alpha_t, \beta_t\right)}\right] \\ - b&\left[\frac{G\left(\frac{1}{b}\middle|\,\alpha_t, \beta_t\right) - G\left(\frac{1}{c}\middle|\,\alpha_t, \beta_t\right)}{1 - G\left(\frac{1}{c}\middle|\,\alpha_t, \beta_t\right)}\right], \qquad \alpha_t > 1.\end{aligned} \tag{3.33}$$

As one can deduce, the pricing formula above will assume the role played by copula functions in 'bottom-up' models. Indeed, equipped with this Gamma-based formula, we can calibrate the model to market quotations to solve for the volatility and correlation parameters of the CLR model developed earlier, and make use thereafter of this best-fit volatility-correlation structure to price CDO derivatives employing the same Gamma-based formula again.[4] However, as we will see just below, the main difference with copulas is that we are able to recover the implied IG distribution for any time date in the future, which allows for building a dynamic setup of the aggregate pool loss.

# 4 Calibration & Numerical Examples

## 4.1 *Pricing CDO Tranches*

Consider a CDO tranche $j$ with attachment/detachment points $(a_j^L, a_j^U)$. Let $N_{j,t}$ be the time $t$-outstanding notional of the CDO tranche $j$ verifying,

$$N_{j,0} = N_0\left(a_j^U - a_j^L\right), \tag{4.1}$$

$$N_{j,t} = N_0\left(a_j^U - a_j^L - \ell_{j,t}\right), \tag{4.2}$$

with $\ell_{j,t}$ is the normalized CCL up to time $t$ over the tranche $j$,

$$\ell_{j,t} = \min\left[\ell_t, a_j^U\right] - \min\left[\ell_t, a_j^L\right]. \tag{4.3}$$

[4] Notice that for general values of the Gamma function parameters, $\alpha_t$,$\beta_t > 0$, the excess credit loss $\mathbf{EL}(b,t)$ as well as the moments of $\ell_t$ could be easily computed numerically using the Gamma's c.d.f., already available in computing software solutions and spreadsheets. In our calibration procedure, we compute these quantities numerically in order to not force the Gamma function's parameters to respect the technical condition $\alpha_t > p$ $(p = 1, 2...)$ required for a closed-form solution. Nevertheless, this condition is met by our model once calibrated to market quotations.

Let $t_u = t_1, t_2, ...t_v$ be the payment dates over the CDO with $t_0 = 0$ and $t_v = T$, where $T$ is the CDO term. Without loss of generality, we use the convention according to which credit losses are suffered at the mid-time of payment periods of the CDO; that is at $\bar{t}_u = (t_{u-1} + t_u)/2$. This is not required to solve for the values of CDO tranches, but it is helpful to simplify notations. Therefore, we can express the time 0-values of the Protection Leg and the Premium Leg of the tranche $j$ as follows:

$$\begin{aligned} V_{j,\mathrm{Protection}}(t_0) &= \sum_{u=1}^{v} D(\bar{t}_u)\mathbb{E}_0^{\mathbb{Q}}\left[N_{j,t_{u-1}} - N_{j,t_u}\right] & (4.4)\\ &= N_0 \sum_{u=1}^{v} D(\bar{t}_u)\mathbb{E}_0^{\mathbb{Q}}\left[\ell_{j,t_u} - \ell_{j,t_{u-1}}\right], & (4.5)\\ V_{j,\mathrm{Premium}}(t_0) &= \tau\bar{s}_j \sum_{u=1}^{v} D(t_u)\mathbb{E}_0^{\mathbb{Q}}\left[N_{j,t_u}\right] + \tfrac{1}{2}\tau\bar{s}_j \sum_{u=1}^{v} D(\bar{t}_u)\mathbb{E}_0^{\mathbb{Q}}\left[N_{j,t_{u-1}} - N_{j,t_u}\right] & (4.6)\\ &= \tau\bar{s}_j N_0 \sum_{u=1}^{v} \left[D(t_u)\left(a_j^U - a_j^L - \mathbb{E}_0^{\mathbb{Q}}\left[\ell_{j,t_u}\right]\right) + \tfrac{1}{2}D(\bar{t}_u)\mathbb{E}_0^{\mathbb{Q}}\left[\ell_{j,t_u} - \ell_{j,t_{u-1}}\right]\right] & (4.7) \end{aligned}$$

where $D(t)$ is the discount factor (the present value of 1\$ to be received at time $t$), $\tau := t_u - t_{u-1}$ is the length of payment periods (typically, $\tau = 0.25$), and $\bar{s}_j$ is the tranche's fair spread fixed at the inception date $t_0$ such that $V_{j,\mathrm{Protection}}(t_0) = V_{j,\mathrm{Premium}}(t_0)$.

By noticing that the spread $\bar{s}_j$ is simply a function of the expected CCL, $\mathbb{E}_0^{\mathbb{Q}}\left[\ell_{j,t_u}\right]$, at the payment dates $t_u = t_1, t_2, ...T$, and making use of the following equation,

$$\mathbb{E}_0^{\mathbb{Q}}\left[\min\left(\ell_t, b\right)\right] = \mathbb{E}_0^{\mathbb{Q}}\left[\ell_t\right] - \mathbf{EL}(b, t), \qquad b \in [0, 1], \tag{4.8}$$

we have

$$\bar{s}_j = \frac{\sum_{u=1}^{v} D(\bar{t}_u)\left(\Pi_j(t_u) - \Pi_j(t_{u-1})\right)}{\tau \sum_{u=1}^{v} \left[D(t_u)\left(a_j^U - a_j^L - \Pi_j(t_u)\right)\right] + \left[\frac{1}{2}D(\bar{t}_u)\left(\Pi_j(t_u) - \Pi_j(t_{u-1})\right)\right]}, \tag{4.9}$$

with,

$$\Pi_j(t) := \mathbf{EL}(a_j^L, t) - \mathbf{EL}(a_j^U, t). \tag{4.10}$$

## 4.2 *Calibration Procedure*

The calibration of the model to market quotations of CDO tranches proceeds as follows. First, we regroup the constituents of the CDS index (i.e., the underlying portfolio of the CDO) into *homogenous groups* based on their CDS credit spreads. Each group will reflect a credit class with specific default risk intensity. This enables us to reduce considerably the number of model parameters to calibrate, without loosing much of market information. For instance, we may use the average CDS spread over the quoted CDS spreads for different terms as a proxy of credit risk intensity. Once this step is achieved, we are able to regroup the $n$ constituents into $q$ groups (the number of groups $q$ could be set equal to 2 or 3) ranging from the lowest to the highest credit risk.[5] Let these groups indexed by $k$ with $k = 1, ...q$. Recalling the explicit Markovian CLR model proposed earlier, we assume that credit entities belong the same group or credit class $k$ will exhibit the same CLR's volatility structure,

$$\sigma_k(t,T) = \bar{\sigma}_k \ e^{-a_k(T-t)}, \tag{4.11}$$

which is equivalent to formulating the SDE of the spot CLR of any reference name $i$ affected to the group $k$ as follows:

$$dx_{i,t} = [\psi_i(t) - a_k x_{i,t}]\, dt + \bar{\sigma}_k dW_i(t), \tag{4.12}$$

$$\psi_i(t) = \frac{\partial f_i(0,t)}{\partial t} + a_k f_i(0,t) + \frac{\bar{\sigma}_k^2}{2a_k}\left(1 - e^{-2a_k t}\right). \tag{4.13}$$

This means that the calibration of the model's volatility structure will require a set of $2q + 1$ parameters: $(\bar{\sigma}_k, a_k)_{k=1,...q}$ and the instantaneous CLRs correlation parameter $\gamma$. Of course, when needed, this number could be reduced either by compacting homogenous groups (i.e., reducing $q$) or either by imposing to one of the CLR diffusion parameters, $(\bar{\sigma}, a)$, to be the same for the $q$ credit classes or groups (a common reversion-speed $a$ seems a natural candidate in that case). The calibration of the model to the term structure of CDS spreads of each reference name $i$ ($i = 1, ...n$) such that the model is arbitrage-free is ensured by making use of the expression $m(T) = \mathbb{E}_0^{\mathbb{Q}}[X(0,T)]$ involving the market credit spread $s^M(T)$. This part of model calibration is easy to do, since we only need to recover the market curves of the $n$ single names' CDS spreads to obtain $(s_i^M(t_1), ..., s_i^M(t_v))_{i=1,...n}$.

Now, we illustrate how the volatility-correlation structure of CLRs is calibrated to CDO market quotations:

**1.** Fixe the parameter $c$ involved in the IG function, which represents the upper bound of the IG distribution support. Depending on the level of CDS-CDO spreads, setting $c$ at 40% or 50% is sufficient to capture the fat-tail of the CLL distribution for any projection time $t = t_1, t_2, ...t_v$ under a 'normal' market environment. Based on our numerical experiences, increasing this value to 100% and best-fitting the model thereafter will not introduce significant changes on the best-fit parameters. In addition, under severe market conditions like those experienced recently, extreme credit losses near the 100% level are most likely priced the market. All of these facts argue the idea to fix this parameter at its maximum arbitrage-free level of 100% before proceeding to the optimization step.

**2.** Initialize the model's $(2q + 1)$ parameters of the $q$ CLRs diffusions and set the link between them and the IG pricing formula of CDO tranches using the first two moments of the normalized pool loss $\ell_t$. Indeed, for any chosen values of the CLRs diffusions parameters, we can invert the first two moments of $\ell_t$ in order to solve for the IG function parameters $(\alpha_t, \beta_t)$. This establishes the fundamental relationship between the CLR model in one hand and the IG pricing formula of

[5] Giesecke and Goldberg (2005) develop an extended 'top-down' approach dealing with a similar high-dimensionality problem and Papageorgiou and Sircar (2008) propose a similar name grouping solution for valuation purposes.

CDO tranches in another hand. To recover the IG parameters $(\alpha_t, \beta_t)_{t=t_1,t_2,...t_v}$ form any chosen parameters $\{(\bar{\sigma}_k, a_k)_{k=1,...q}, \gamma\}$ of CLRs diffusions, we solve the system

$$\begin{aligned}
\mathbb{E}_0^{\mathbb{Q}}\left[\ell_t^{p=1}\right] &= \frac{1}{\beta_t(\alpha_t - 1)}\left[\frac{1 - G\left(\frac{1}{c}\middle|\,\alpha_t - 1, \beta_t\right)}{1 - G\left(\frac{1}{c}\middle|\,\alpha_t, \beta_t\right)}\right] && (4.14)\\
&= M_t^{p=1}((\bar{\sigma}_k, a_k)_k, \gamma; s_i^M(t)_i), && (4.15)\\
\mathbb{E}_0^{\mathbb{Q}}\left[\ell_t^{p=2}\right] &= \frac{1}{\beta_t^2(\alpha_t - 1)(\alpha_t - 2)}\left[\frac{1 - G\left(\frac{1}{c}\middle|\,\alpha_t - 2, \beta_t\right)}{1 - G\left(\frac{1}{c}\middle|\,\alpha_t, \beta_t\right)}\right] && (4.16)\\
&= M_t^{p=2}((\bar{\sigma}_k, a_k)_k, \gamma; s_i^M(t)_i), && (4.17)
\end{aligned}$$

where

$$\begin{aligned}
M_t^{p=1}((\bar{\sigma}_k, a_k)_k, \gamma; s_i^M(t)_i) &= \frac{1}{N_0}\sum_{i=1}^{n}\bar{L}_i\left(t\ \middle|(\bar{\sigma}_k, a_k)_k, \gamma; s_i^M(t)_i\right),\\
M_t^{p=2}((\bar{\sigma}_k, a_k)_k, \gamma; s_i^M(t)_i) &= \frac{1}{N_0^2}\sum_{i=1}^{n}\bar{L}_{ii}\left(t\ \middle|(\bar{\sigma}_k, a_k)_k, \gamma; s_i^M(t)_i\right)\\
&\quad + \frac{1}{N_0^2}\sum_{i=1}^{n}\sum_{j=1(i\neq j)}^{n}\bar{L}_{ij}\left(t\ \middle|(\bar{\sigma}_k, a_k)_k, \gamma; s_i^M(t)_i\right).
\end{aligned}$$

This non-linear system could be easily solved at each iteration of the optimization procedure (i.e., the pricing-error minimization task) using the bivariate Newton-Raphson technique. To accelerate numerical resolution, the initial solution 'guess' could be set equal to the closed-form solution we get under the assumption of standard IG distribution (i.e., $c = \infty$) given by,

$$\begin{aligned}
\hat{\alpha}_t &= \frac{2M_t^{p=2} - \left(M_t^{p=1}\right)^2}{M_t^{p=2} - \left(M_t^{p=1}\right)^2}, && (4.18)\\
\hat{\beta}_t &= \frac{M_t^{p=2} - \left(M_t^{p=1}\right)^2}{M_t^{p=1}M_t^{p=2}}. && (4.19)
\end{aligned}$$

Based on our numerical tests, this closed-form solution is very close to the true solution (less than $\pm 5\%$ of relative error).

**3.** Best fit the parameters $\{(\bar{\sigma}_k, a_k)_{k=1,...q}, \gamma\}$ and recover the associated best-fit IG parameters $(\alpha_t, \beta_t)_{t=t_1,t_2,...t_v}$ determining the market implied distribution of the aggregate pool loss at any projection time $t = t_1, t_2, ...t_v$.

### 4.3 *Remarks*

**i)**.It is worthwhile to note that the distribution of the CCL as the sum of $n$ correlated lognormals does not *exactly* behave as the IG distribution.[6] The model is only built on the realistic assumption that the IG distribution is a good approximate distribution of the CCL under the LCCL framework. To what extent the IG distribution deviates from the true (but unknown) distribution of the CCL (even though these deviations must have very small size) should not be critical in itself whenever we make use of the IG based valuation formula for pricing purposes. In other words, once we operate under the IG valuation framework, calibration and pricing are consistent with each other at the aggregate pool level, which is sufficient enough for the model to be coherent and works correctly. Even the matching of single names' CDS spreads ensured at the 'down' level of the LCCL model is consistent with the IG valuation framework, since the IG-based mean $M_t^1(\alpha_t, \beta_t)$ of the pool's CCL (involving the $n$ single names' CDS spreads) is calibrated to equalize the LCCL-based mean $M_t^1((\bar{\sigma}_k, a_k)_k, \gamma; s_i^M(t)_i)$. Similarly, the eventual *error* caused by the IG approximation should not affect the consistency of the pricing of complex CDO derivatives that would require some numerical implementation of the model, whenever this is done at the aggregate pool loss level (i.e., under the IG framework), as we will show in the end of the paper. In this sense, the IG approximate solution could be viewed as an equivalent pricing tool to the copulas-based approximate formula in 'bottom-up' approach, but with the main difference that the IG model allows for modeling the dynamics of the credit pool loss, while standard 'bottom-up' models do not.

**ii)**.With respect to the fundamental relationship between the CLR model and the IG pricing formula, one would think that we do not need to build our pricing model based on the CLR diffusions, but simply by best-fitting directly the IG parameters $(\alpha_t, \beta_t)_{t=t_1,t_2,\ldots t_v}$ to the observed spreads of CDO tranches. This avenue would make the model behind of correlated single names' CLRs useless, but it is in fact not feasible. When doing so, the optimization procedure used for calibration will attribute arbitrary values to the IG parameters $(\alpha_t, \beta_t)$ corresponding to projection times $t \leq t_{v-1}$ prior to the CDO maturity, and thus will focus the best-fit on the last pair $(\alpha_{t_v}, \beta_{t_v})$ associated with the maturity date $t_v$ of the CDO. Although it may lead to a good fit of market quotations, this calibration outcome clearly results in unrealistic implications in terms of the implied dynamics of the credit pool loss and the underlying single names' defaults. Indeed, we should see the IG pricing formula as a pricing tool of the aggregate credit loss only, since it cannot substitute the CLR as a tool for modeling defaults of single names in a satisfactory manner.

### 4.4 *Examples*

We now calibrate the LCCL model to two different market quotations of CDO, both written on the iTraxx Europe CDS index. The first quotation numbers are those used by Bennani (2005) and refer to market spreads as of $17^{th}$ October 2005. This choice aims to show the relative superiority of the LCCL model to fit the default correlation skew with comparison to standard 'top-down' models. The second quotation numbers are relatively recent (market spreads as of $10^{th}$ April 2008), which

[6] The term 'exactly' is meant to be here that both the IG distribution constitutes the *true* distribution of the CCL.

allows us to evaluate the model under extreme market conditions, such as those related to the current crisis of subprime.

#### 4.4.1 *Example* 1 ($17^{th}$ *October* 2005):

**Figure 1** illustrates the market 5 years-CDO spreads and the best-fit spreads of both the LCCL model (the LCCL model implemented allows for tow credit classes (i.e., $q = 2$)) and the 'top-down' lognormal model of Bennani (2005). We decided to report the fitting results based on the CDO spreads rather than the implied base correlations. This is motivated by the fact the base correlation numbers are less optimal to show significant differences between the market spreads of CDO tranches (both their average level and skew) and the model predicted spreads (for any model).

As we can see, our LCCL model performs very well to fit both the average level of market CDO spreads and their skewed shape, while the best-fit of the 'top-down' model is relatively poor, particularly for the skew of market quotations. The values of these spreads are summarized in **Table 1**. As we have stated earlier, this superiority of the LCCL model is mainly explained by the allowance for single names-based forward loss dynamics and correlation between them rather than directly modeling the aggregate forward loss of the CDS index. Technically, the allowance for correlated forward losses of single names permits the use of the IG function as a distribution of the pool loss, which yields better fit results than the 'top-down' lognormal model of Bennani.

It is worthwhile to note that the Bennani's 'top-down' model is flexible and could allow for jump component, which would enhance its fitting quality. However, we show here that conditional on the use of continuous CLR processes only, our LCCL model performs relatively better. We should also point out that our LCCL model is calibrated for any maturity horizon to the entire set of CDO tranches and fits well the skew formed from the market spreads of these tranches. In fact, this is not allowed by all 'top-down' models. For example, the periodic impulses model of Hull and White (2007) −similarly to the Black-Scholes model for stock options failing to capture the volatility smile− has to be calibrated to each CDO tranche at once in order to fit well.

**Figure 2-A** plots the market implied gamma p.d.f. $g\left(\frac{1}{\ell_t}\middle|\,\alpha_t,\beta_t\right)$ corresponding to the LCCL model's best -fit parameters for different time horizons $t = 1, 2, ..., 5$ years (as inferred from the fitted LCCL model to the 5 years-CDO market spreads). For the same time horizons, **Figure 2-B** and **Figure 2-C** illustrate the market implied IG p.d.f. $\hat{g}_I\left(\ell_t|\,\alpha_t,\beta_t;c\right) = \left(g_I\left(\ell_t|\,\alpha_t,\beta_t\right)/G_I\left(c|\,\alpha_t,\beta_t\right)\right)$ and the market implied IG c.d.f. $G_I\left(\ell_t|\,\alpha_t,\beta_t\right)$ of the pool loss, respectively. As expected, the unconstrained gamma p.d.f. shows a regular skewness for all the considered projection times, while the IG p.d.f. used in the valuation of CDO spreads exhibits a skewness highly decreasing with the loss projection horizon and a fat-tail loss increasing with time.

**Figure 3** plots stream lines representing the expected excess credit losses for different loss triggers (corresponding to the attachment points of CDO tranches) over a time interval of 5 years (as inferred from the fitted LCCL model to the 5 years-CDO market spreads). In conformity with the skew of CDO spreads, the credit loss in excess of the first trigger of 3% exhibits the higher slope with comparison to the subsequent excess losses. This model feature is important, since it is responsible for whether or not we are able to replicate the average level of market CDO spreads while being allowed to reproduce the same shape of default correlation skew.

Now, with regard to the CLRs model behind the IG-pool loss, **Figure 4-A** plots the market implied credit correlation over a time interval of 10 years as inferred from the LCCL model fitted to the 5 years-CDO market spreads. The figure illustrates the instantaneous correlation function of CLRs, $\rho(t) = 1 - e^{-\gamma t}$, as well as the correlation function of credit losses $\rho_{ij}^{L}(t)$ for two pairs of reference names selected from the CDS index. The first pair combines two names belong to the first group of high credit quality ($k = 1$), while the second regroups two names belong to the second group of low credit quality ($k = 2$). Remember here that the level of credit loss correlation depends on the CDS spreads and the CLRs processes of the involved reference names, which makes this correlation intensity specific to each pair considered. We see that even though the instantaneous correlation is high and extremely concave (starts from 22% at 0.50 year to reach 80% at the 3 years term), the credit loss correlations maintain a reasonable level and increases slowly over time. This is a direct consequence of the fact that credit losses result from the CLRs function $X(0,T)$ rather than the spot CLR itself. **Figure 4-B** plots the volatility structure of CLRs represented by the square-root of the variance $\upsilon_k(0,t)$ of the CLR functions $X_k(0,t)$ for the two credit classes $k = 1, 2$ (as inferred from the fitted LCCL model to the 5 years-CDO market spreads). We observe that the level of CLR volatility is largely dependent on the credit quality of the considered group of constituents/single names, which is a clear indication of the relevance of the single name-based approach we propose to model forward credit losses.

### 4.4.2 *Example* 2 ($10^{th}$ *April* 2008):

The market CDO spreads as of $10^{th}$ April 2008 are extremely large with comparison to those of the last calibration example, due to the credit crisis context behind. As one may expect, this severe market environment is challenging to any model. The calibration results are illustrated in **Figure 5**. Similarly to the previous calibration example, the LCCL model was implemented using tow credit classes (i.e., $q = 2$). We see that the model performs less than in the last regular market example of 2005. Particularly, we noticed after multiple tests that the model has a difficulty to replicate the two segments of the CDO curve, the first formed by the 3-6% and 6-9% tranches and the second segment formed by the remaining higher protection tranches. We are able to enhance the replication of only one of these two segments of the CDO curve, but not the whole curve at once. We conclude that under these extreme market conditions, the model cannot generate a fat-tail large enough to capture the spreads of higher-protection tranches (i.e., the second segment of the CDO curve), while preserving a good skewness of the aggregate loss distribution matching the spread of the first mezzanine tranche 3-6%. This clearly is not attributable to the Gaussian distribution of CLRs, since we must recall that the pool loss upon which CDO tranches are priced follows the IG distribution in our LCCL model. It is also known that the gamma functions are among the most skewed distributions. To replicate these market spreads, therefore, we have to consider further extensions of the LCCL-IG model above, which is our purpose in the next section.

# 5 Super-Senior Pricing under Stressed Market: The LCCL-Fat-Tail Model

We propose here an extension of the basic LCCL model introduced earlier aiming to improve its capacity of fitting market quotations of CDO spreads. Namely, we try to correct for the lack of fat-tail generated by the IG distribution to enhance the capacity of the model to replicate the skew of market CDO spreads. As we have seen upon the calibration of the basic LCCL model to the market quotations of $10^{th}$ April 2008 (*Example* 2), the IG function does not allow at the same time for a sufficient fat-tail permitting both: i) the matching of the market spreads of higher-protection tranches, and ii) the prediction of a spread for the first 3-6% tranche in line with the market. Indeed, the model predicts spreads that are higher than the market level for the 3-6% tranche when the fit of higher-protection tranches is relatively good. In addition, the model predicted spread for the super senior 22-100% tranche is largely below the market spread. Clearly, this is a fat-tail problem in the sense that the IG function cannot accommodate for extreme loss scenarios. Another indication of a lack of fat-tail is that the enhancement of the fit of higher-protection tranches comes at the cost of predicting a deeply high spread for the first 3-6% tranche; a scaling problem due to the interdependence between skewness and fat-tail.

To resolve this problem, we propose here a slight alteration of the IG function representing the distribution of the aggregate pool loss. Indeed, rather than acting backward on the features of the single names' CLRs processes, we directly introduce modifications on the aggregate pool loss distribution. It follows that we will work 'outside' the arbitrage-free CLR model of reference names. We discuss later this point more in details.

Formally, the 'LCCL-fat-tail' model consists of making the assumption that the pool loss $\ell_t$, as projected for time $t$, is better described by a mixture of the IG function used earlier and an extra 'fat-tail leg' as follows:

$$\mathbb{Q}\left[\ell_t \in d\ell\right] = \begin{cases} \hat{g}_I\left(\ell \middle| \alpha_t, \beta_t; c=1\right) d\ell & \text{for} \quad \ell \in \left[0, \bar{\ell}_t\right], \\ p_t(\ell) d\ell & \text{for} \quad \ell \in \left[\bar{\ell}_t, 1\right], \end{cases} \tag{5.1}$$

where $p_t(\ell)$ represents a given p.d.f. artificially mixed to the IG function and $\bar{\ell}_t$ is the loss trigger at which the aggregate pool loss p.d.f. switches from the IG function to the 'extra fat-tail leg' $p_t(\ell)$. The idea is that by controlling the choice of this portion of the credit loss p.d.f., $p_t(.)$, we are able to enhance the capacity of the model to fit higher-protection tranches, while preserving the skewness of the original IG function to fit well lower-protection tranches; thus reproducing as much as possible the observed skew of market CDO spreads.

Let's assume a simple linear function for the fat-tail leg of the form,

$$p_t(\ell) := \psi_{t,0} + \psi_{t,1}\ell, \quad \psi_{t,0} > 0, \quad \psi_{t,1} \leq 0.$$

Using the arbitrage-free condition $\ell_t \in [0,1]$ and the total probability law together, the function $p_t(.)$ must verify for any projection horizon $t > 0$ the following equality,

$$\int_0^{\bar{\ell}_t} \hat{g}_I\left(\ell | \alpha_t, \beta_t; 1\right) d\ell + \int_{\bar{\ell}_t}^1 p_t(\ell) d\ell = 1, \tag{5.2}$$

which is equivalent to,

$$\int_{\bar{\ell}_t}^1 p_t(\ell) d\ell = 1 - \left[\frac{1 - G\left(1/\bar{\ell}_t | \alpha_t, \beta_t\right)}{1 - G\left(1 | \alpha_t, \beta_t\right)}\right]. \tag{5.3}$$

Then, imposing the continuity of the p.d.f. $\mathbb{Q}[\ell_t \in d\ell]$ around the switching point $\bar{\ell}_t$ yields the following conditions after making use of the last equality and the linear form of $p_t(.)$:

$$\left[\left(\hat{g}_I\left(\bar{\ell}_t | \alpha_t, \beta_t; 1\right) - \psi_{t,1}\bar{\ell}_t\right)\left(1 - \bar{\ell}_t\right)\right] + \frac{\psi_{t,1}}{2}\left(1 - \bar{\ell}_t^2\right) = 1 - \left[\frac{1 - G\left(1/\bar{\ell}_t | \alpha_t, \beta_t\right)}{1 - G\left(1 | \alpha_t, \beta_t\right)}\right], \tag{5.4}$$

$$\psi_{t,0} = \hat{g}_I\left(\bar{\ell}_t | \alpha_t, \beta_t; 1\right) - \psi_{t,1}\bar{\ell}_t. \tag{5.5}$$

Finally, ensuring that the fat-tail leg function $p_t(.)$ will never be negative requires the boundary constraint $p_t(1) \geq 0$, which reads

$$\psi_{t,1} \geq -\left(1 - \bar{\ell}_t\right)^{-1} \hat{g}_I\left(\bar{\ell}_t | \alpha_t, \beta_t; 1\right). \tag{5.6}$$

By solving for $\left(\psi_{t,0}, \psi_{t,1}\right)$ after fixing the loss trigger $\bar{\ell}_t$, we are able to price the excess credit loss accordingly and then solve for the CDO spreads. The numerical implementation we adopt for the LCCL-fat-tail model above is as follows. We start in a first step by solving for the original basic LCCL model. To ensure a good final fit, we orient the fitting quality to be concentrated around the lower-protection tranches (i.e., the first tranches) by applying a weighted-average of squared-relative errors as objective function to minimize. This enables us to let the original basic LCCL model working to provide a better fit for lower-protection tranches, while yielding the worst fitting performances for the higher-protection tranches. Once the CLRs processes' best-fit parameters are obtained, we move in a second step to the fat-tail component in order to enhance the fitting quality of the higher-protection tranches. We start by fixing for each projection horizon $t_u = t_1, t_2, ...T$ a given loss trigger $\bar{\ell}_{t_u}$. Based on our numerical experiences, we rarely need to apply the extra fat-tail extension for short-term projection times less than 4 years. Only projection horizons exceeding 4 years generally require the introduction of a fat-tail component, which is quite intuitive given the economic meaning behind. For these projection dates, we apply a loss trigger that increases with the projection time $t_u$, which preserves the time-homogeneity of the credit loss and ensures a consistency between the fat-tail component and the arbitrage-free condition of time-increasing CLL. We fix these triggers exogenously and then solve for the best linear fit $\left(\psi_{t_u,0}, \psi_{t_u,1}\right)$ of the extra fat-tail $p_{t_u}(.)$. Solving for the loss triggers simultaneously with the linear fat-tail leg parameters $\left(\psi_{t_u,0}, \psi_{t_u,1}\right)$ is possible, but this imposes some challenges in a forward-looking context when using the model to price CDO derivatives. The reason is that in contrast to the parameters $(\alpha_t, \beta_t)$ of the IG function that can be determined in a forward-looking context (i.e., for $t > T$) in function of the bet-fit CLRs processes using the first two moments of the aggregate loss, the loss trigger $\bar{\ell}_t$ cannot be recovered from the best-fit CLRs processes. Nevertheless, fixing this trigger initially or searching for its best-fit value yields virtually the same final best-fit quality. The way by which we fix these triggers initially is simple: we optimize both the 4 year-loss trigger $\bar{\ell}_{t_u=4}$ and the constant increment we add to recover the next triggers corresponding to the subsequent time dates $t_u = 4+\tau, 4+2\tau....$ We do this once we solve for the best-fit parameters $\left(\psi_{t_u,0}, \psi_{t_u,1}\right)$ for each $t_u \geq 4$. This enables us to forecast the loss triggers for projection times exceeding $T$ (i.e., in a forward-looking context) accordingly to the best fit of time 0-market spreads by applying the same constant increment.

The LCCL-fat-tail model is implemented as described above to fit the market quotations of $10^{th}$ April 2008 (*Example* 2). The best fits provided by the basic LCCL model and the 'extra fat-tail' model are illustrated in **Figure 5**. As one can notice, the fitting quality of the fat-tail model is better than which provided by the basic LCCL model. We see that the fit of this model is good for both the lower and the higher protection tranches. In particular, the model predicted spreads for the higher protection tranches (those requiring the extra fat-tail to replicate the market implied extreme loss) are more line with the observed market spreads than which predicted earlier under the basic LCCL model. **Figure 6-A** and **Figure 6-B** plot, respectively, the probabilities $\mathbb{Q}\left[\ell_t > 12\%\right]$ and $\mathbb{Q}\left[\ell_t > 22\%\right]$ as resulting from the basic LCCL model and the LCCL-fat-tail model (i.e., the mixture of the IG function and the fat-tail component) for the projection time $t = 5$. Finally, **Table 2** summarizes the best fit values of CDO spreads as of date $10^{th}$ April 2008 (*Example* 2) for both the basic LCCL model and the LCCL-fat-tail model.

Observe that by imposing the mixture of the IG function and the fat-tail leg as a p.d.f. of the credit pool loss, we are no longer allowed to make the assumption that the CLRs processes of the single names are given by the Gaussian process of the basic LCCL model. In other words, we are now in a situation where the individual CLR processes of the single names are unknown. Only the IG component of the pool loss's p.d.f. is generated by the LCCL model, while the mixed p.d.f. cannot be attributed to the LCCL model alone. Recalling the HJM-properties of the LCCL model, this imposes to us to work 'outside' the arbitrage-free framework used earlier. As a consequence, the matching of the CDS spreads curves of the single names is no longer valid in full, since the *true* CLR process is unknown. This is the cost of enhancing the best-fit quality of the model. However, the added fat-tail component will never alter the whole probabilistic properties of the LCCL model. In addition, we can offset the drawback of the *uncertain* matching of CDS spreads of the single names by targeting instead the matching of the CDS index spreads. This can be easily handled by restricting the pool loss's p.d.f. to replicate the curve of the CDS index spreads.

# 6 Model Applications

## 6.1 *Pricing CDO Derivatives*

To price European-style CDO derivatives such as Forward Starting CDO's and CDO Options, one needs to compute quantities involving forward expected credit losses, where the maturity date of the derivative usually serves as the time date at which these forward expectations will be taken. This could be fairly handled under our LCCL model as well as its extended forms by making use of straightforward numerical integration over quantities involving the IG function quite similar to the expected excess loss given by (3.33).

For the model to be applied to price CDO derivatives with early-exercise-style and/or loss trigger features such as Leveraged Super Senior Notes and Bermudan CDO Options, we need to dispose of a discrete representation of the IG distribution model in order to make use of Monte Carlo-Least Squares algorithms. Of course, one can alternatively resort to the 'down' level by propagating the single names' LCCL trees or Monte Carlo paths using the homogenous groups' classification technique employed earlier in the model calibration procedure to reduce the high dimension problem. But we believe that methods based on the IG density of the aggregate pool loss can deliver higher computational performances given their reduced-form nature and their capacity to efficiently summarize the high-dimension universe of the single names' defaults. This numerical extension of the model is discussed just below.

## 6.2 *Monte Carlo Simulation of the Aggregate Pool Loss*

### 6.2.1 *The Discrete Simulation Procedure*

Usually, Monte Carlo simulations of standard Itô diffusions are performed using either discrete schemes of stochastic differential equations or the exact method of the transition density. When dealing with different processes, such as the discontinuous Gamma-Lévy process used by Jäckel (2008), a known transition density could be very helpful to perform simulations. In our case, however, the IG model is calibrated to discrete points over time corresponding to the CDO payment dates $t_u = t_1, t_2, ...T$, which means that the IG model developed here acts as a *pricer* of the market-implied distribution of the pool loss at the given payment dates rather than describing a transition rate for this loss. To simulate paths of the pool loss, we need therefore to perform discrete simulations at each time date and recovering the path from the simulated points by respecting the nonnegative increment property of the pool loss dynamics. To generate pool loss sample paths at these discrete points over the time space, we have first to invert the calibrated c.d.f. $F\left(a|\,\alpha_t,\beta_t\right) \equiv \mathbb{Q}\left[\ell_t \leq a\right]$ (at $t_u = t_1, t_2, ...T$) using stratified uniform samples. Then, for each original sample path $i$, we check whether the nonnegative increment property $\ell^i_{t_u} \geq \ell^i_{t_{u-1}}$ is satisfied by the path $i$. In the negative case, a uniform resampling is required. Letting $W_t\left(a\right) \equiv F^{-1}\left(a|\,\alpha_t,\beta_t\right)$ we have $\ell^i_t = W_t\left(u^i_t\right)$, where $u^i_t \sim$ uniform (0,1). Thus, resampling simply means that we have to sample uniformally respecting $u^i_{t_u} < F\left(W_{t_{u-1}}\left(u^i_{t_{u-1}}\right)\middle|\,\alpha_{t_u},\beta_{t_u}\right)$.[7] Whenever the bounded uniform $u^i_{t_u}$ is randomly generated, no bias will be introduced into the size distribution of the loss increment $\ell^i_{t_u} - \ell^i_{t_{u-1}}$. This is confirmed in **Figure 7-A**, where the time series of the average level of the Monte Carlo paths of the pool loss (over a sample of 50 paths and using the stratified uniform sampling technique) is illustrated against the exact expected loss value $\mathbb{E}^{\mathbb{Q}}_0\left[\ell_t\right]$, both referring to the basic LCCL model of *Example* 1 (calibrated to the 5 years-CDO market spreads as of date $17^{th}$ *October*

[7] Generally, the calibrated IG distributions allow by themselves for an intrinsic increment of loss (this intrinsic increment, as shown by **Figures 2-A** and **2-B**, is well captured by the displacement of the loss's p.d.f. once we move forward the projection time $t_u$), so that a resampling procedure is not necessary for most of the originally generated points.

2005). We see, indeed, that the two stream lines are virtually the same, which argues the use of the proposed discrete simulation procedure.

**Figure 7-B** plots the generated sample of 50 Monte Carlo paths of the pool loss based on which we have computed the average pool loss level illustrated in Figure 7-A. We observe that, in contrast to Brownian motions and their diffusions, the pool loss paths generated by our IG distribution model exhibit nonnegative increments. We also notice that the generated paths are smooth, which is different from the pure Jump process. This property is mainly due to the IG distribution of the loss size as well as the enhanced sample uniformity employed here. In fact, numerical experiences (not reported here) show that the pool loss paths exhibit a pronounced discontinuity illustrated by large intervals of constancy, when standard (*naïve*) sampling is used instead of the stratified uniform sampling technique. Finally, to evaluate the accuracy of the simulation method above for CDO derivatives pricing purposes, we compare the Monte Carlo estimator of the excess credit loss (the Monte Carlo sample is about 5000 generated paths of the pool loss) for the different attachment points (3%-6%-9%-12%-22%) with the corresponding exact values given by the formula of $\mathbf{EL}(a,t)$. **Figure 8** shows the obtained pricing results over a time horizon of 5 years under the basic LCCL model of *Example* 1 (the excess losses are priced at different time projection dates ranging from 0.5, 1, 1.5, ... 5 years). As one can see, the Monte Carlo prices are very close to the exact values, which provides a clear support for the accuracy of the discrete simulation procedure described above.

### 6.2.2 *The Gamma-Bridge Method*

Notice that an alternative simulation approach resulting in pool loss paths propagated at a time increment smaller than the CDO's payment period $\tau$ would consist of making use of a bridging technique, similar to the well-known Brownian-bridge technique, and a local approximation by the Gamma-Lévy process. This method works as follows. We first solve for each time interval $[t_u, t_{u+1}]$ between two subsequent payment dates for the scaling and shape parameters defining a Gamma-Lévy process matching the Gamma distributed variable $S(t) \triangleq \ell_t^{-1}$ for $t = t_u, t_{u+1}$ (ignoring truncation of loss by 1). This is possible by solving for a simple linear system involving the IG parameters $\{(\alpha_t, \beta_t)\,; t = t_u, t_{u+1}\}$. Then based on the symmetric Beta law of the ratio $S(t/2)/S(t)$ (see Dufresne, Gerber and Shiu (1991) for details), one can simulate recursively the variables $S(t_i)$ for $t_i \in [t_u, t_{u+1}]$ starting from the simulated variable $\ell_{t_{u+1}}^{-1} := S(t_{u+1})$. Repeating this procedure for each time interval $[t_u, t_{u+1}]$ by going forward over time, we end up with highly populated paths, where the inverted Gamma-Lévy processes locally calibrated to the IG distribution parameters at the extremities of each considered interval has served as a bridge to propagate paths between each pair of two consecutive payment dates. This solution, however, requires a careful manipulation and its relative benefits are not obvious in all cases. Indeed, since the pool loss at the payment dates is still simulated based on the IG distribution model, this Gamma-bridge technique is only helpful for pricing products that depend on the trajectory of the pool loss. Bermudan CDO options, for example, do not require such a refined simulation and can be simply priced using the simpler discrete simulation procedure described above.

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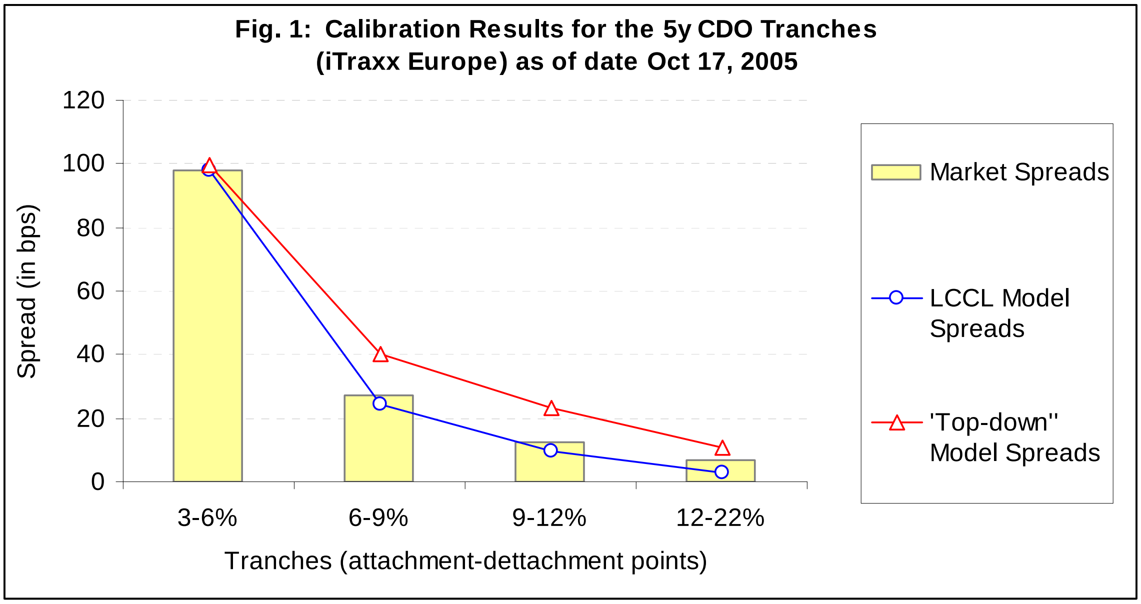
Fig. 1: Calibration Results for the 5y CDO Tranches
(iTraxx Europe) as of date Oct 17, 2005
Spread (in bps)
120
100
80
60
40
20
0
3-6%
6-9%
9-12%
12-22%
Tranches (attachment-dettachment points)
Market Spreads
LCCL Model Spreads
'Top-down'' Model Spreads

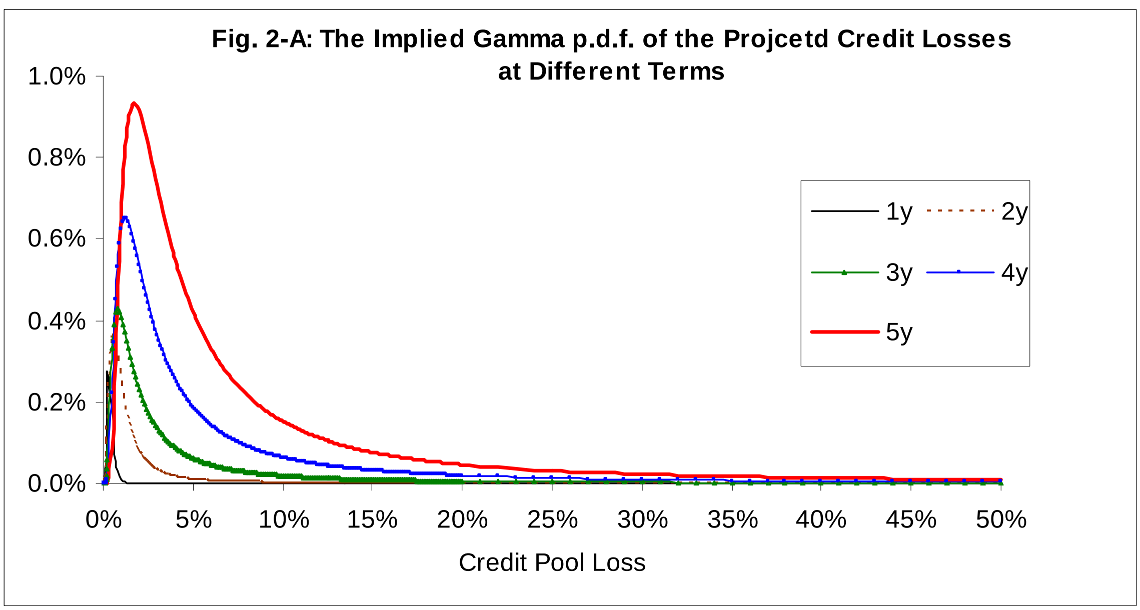
Fig. 2-A: The Implied Gamma p.d.f. of the Projcetd Credit Losses at Different Terms
1.0%
0.8%
0.6%
0.4%
0.2%
0.0%
0%
5%
10%
15%
20%
25%
30%
35%
40%
45%
50%
Credit Pool Loss
1y
2y
3y
4y
5y

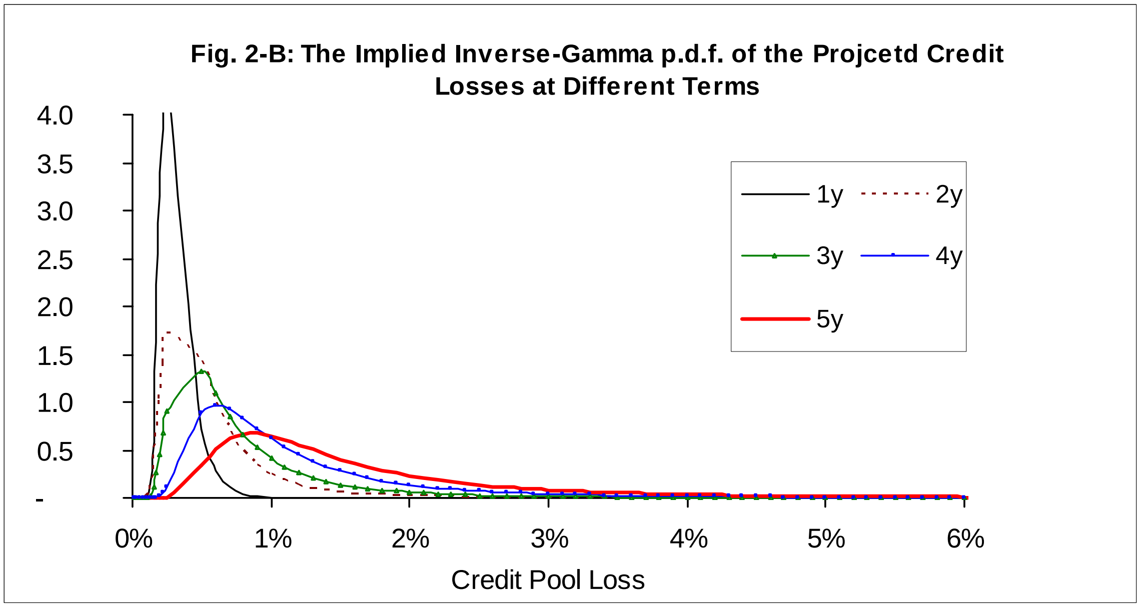

Fig. 2-B: The Implied Inverse-Gamma p.d.f. of the Projcetd Credit Losses at Different Terms
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
-
0%
1%
2%
3%
4%
5%
6%
Credit Pool Loss
1y
2y
3y
4y
5y


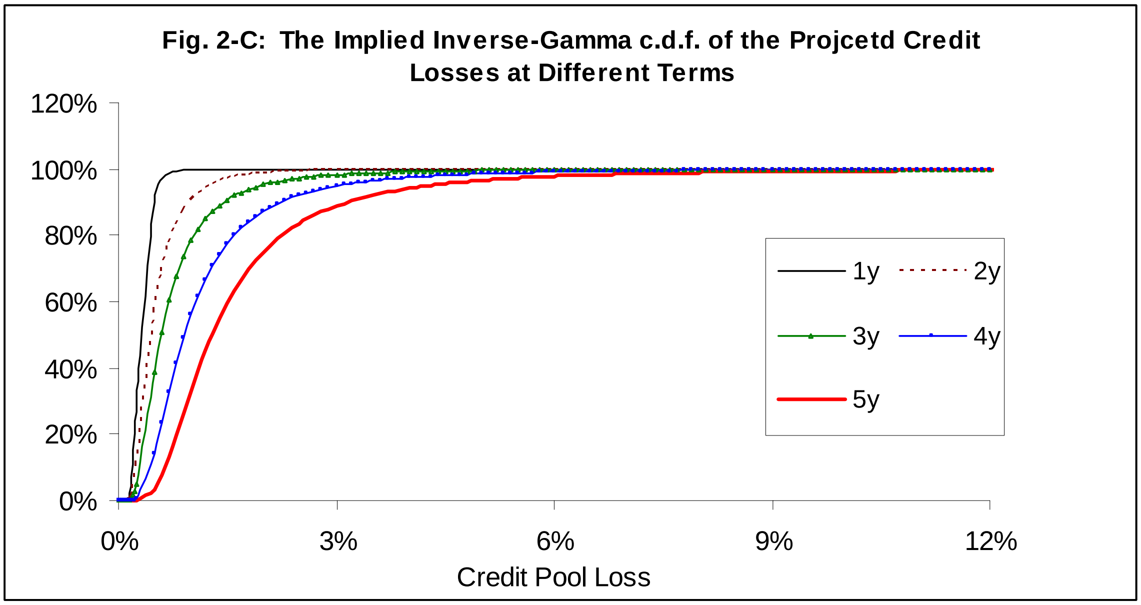

Fig. 2-C: The Implied Inverse-Gamma c.d.f. of the Projcetd Credit Losses at Different Terms
120%
100%
80%
60%
40%
20%
0%
0%
3%
6%
9%
12%
Credit Pool Loss
1y
2y
3y
4y
5y

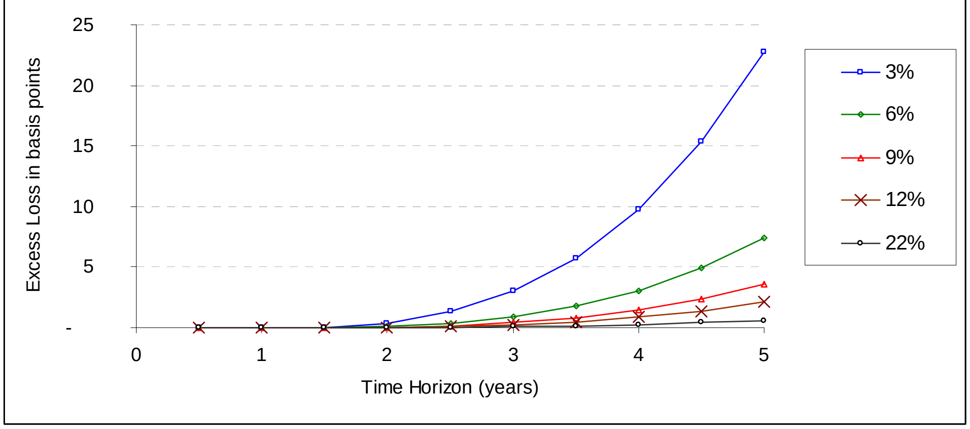


**Fig. 3: Excess Credit Losses (in basis points) as Implied form the LCCL Model**

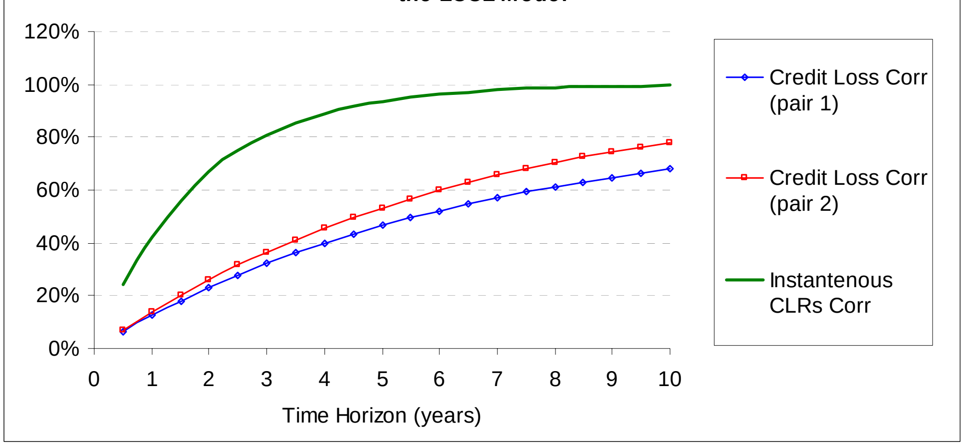


**Fig. 4-A: The Credit Loss Correlation Structure as Implied from the LCCL Model**

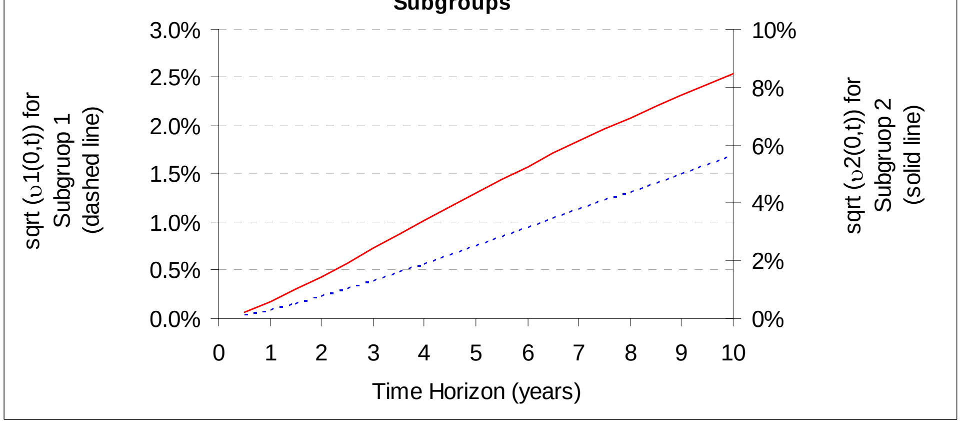


**Fig. 4-B: Square-root of the Variance Term υ(0,t) of the CLR Function X(0,t) as Implied from the LCCL Model for the Two Homogenous Subgroups**

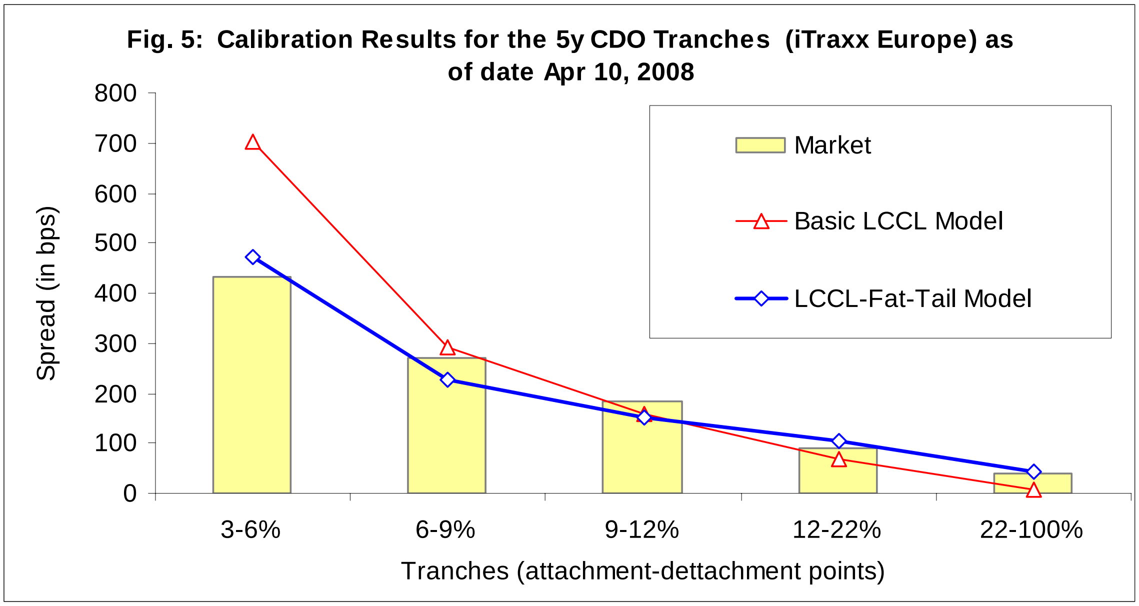
Fig. 5: Calibration Results for the 5y CDO Tranches (iTraxx Europe) as of date Apr 10, 2008
800
700
600
500
400
300
200
100
0
Spread (in bps)
Market
Basic LCCL Model
LCCL-Fat-Tail Model
3-6%
6-9%
9-12%
12-22%
22-100%
Tranches (attachment-dettachment points)

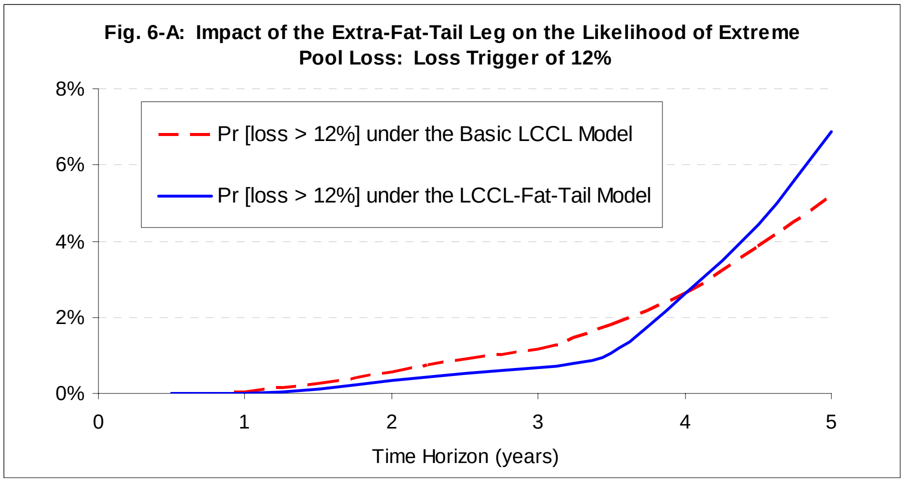
Fig. 6-A: Impact of the Extra-Fat-Tail Leg on the Likelihood of Extreme Pool Loss: Loss Trigger of 12%
8%
6%
4%
2%
0%
Pr [loss > 12%] under the Basic LCCL Model
Pr [loss > 12%] under the LCCL-Fat-Tail Model
0
1
2
3
4
5
Time Horizon (years)

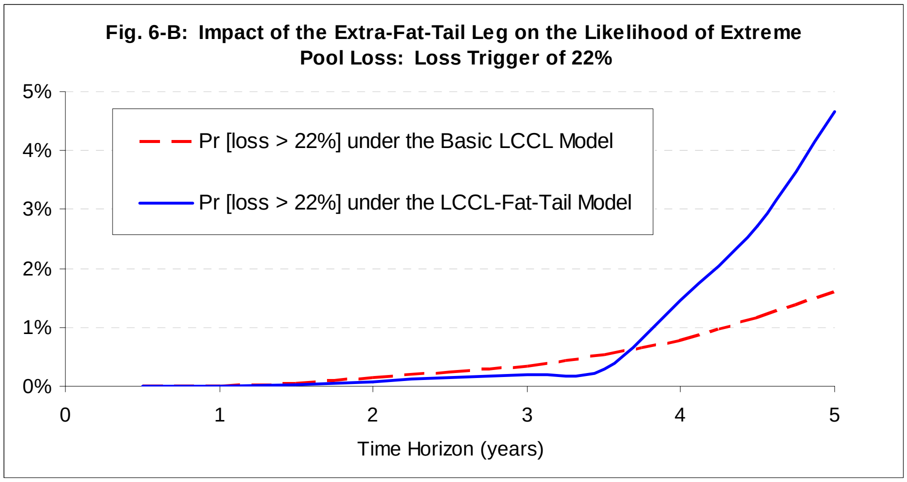
Fig. 6-B: Impact of the Extra-Fat-Tail Leg on the Likelihood of Extreme Pool Loss: Loss Trigger of 22%
5%
4%
3%
2%
1%
0%
Pr [loss > 22%] under the Basic LCCL Model
Pr [loss > 22%] under the LCCL-Fat-Tail Model
0
1
2
3
4
5
Time Horizon (years)

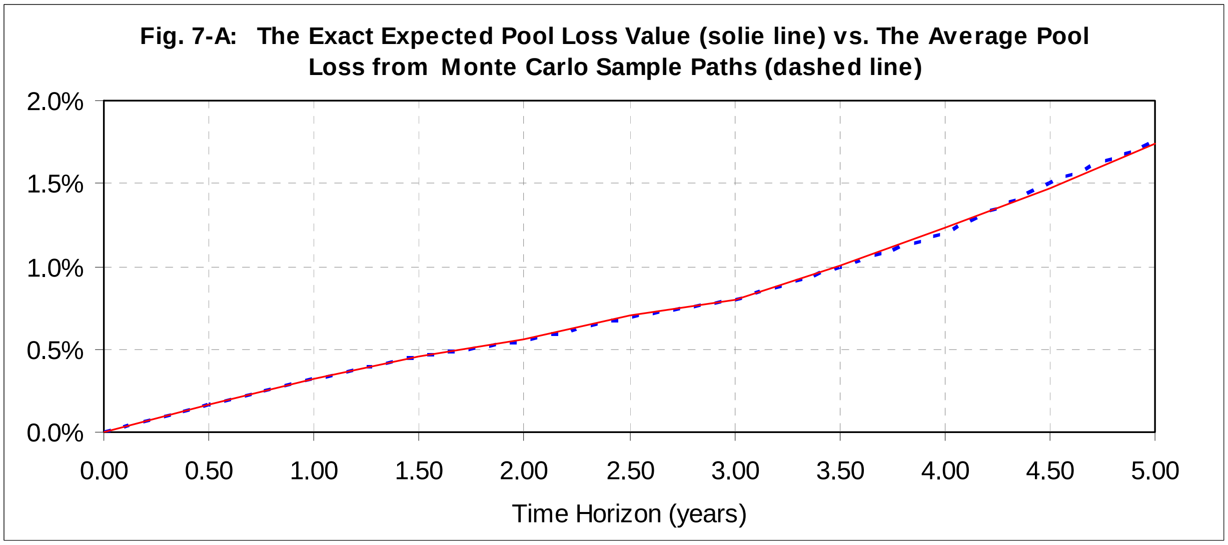

Fig. 7-A: The Exact Expected Pool Loss Value (solie line) vs. The Average Pool Loss from Monte Carlo Sample Paths (dashed line)
2.0%
1.5%
1.0%
0.5%
0.0%
0.00
0.50
1.00
1.50
2.00
2.50
3.00
3.50
4.00
4.50
5.00
Time Horizon (years)


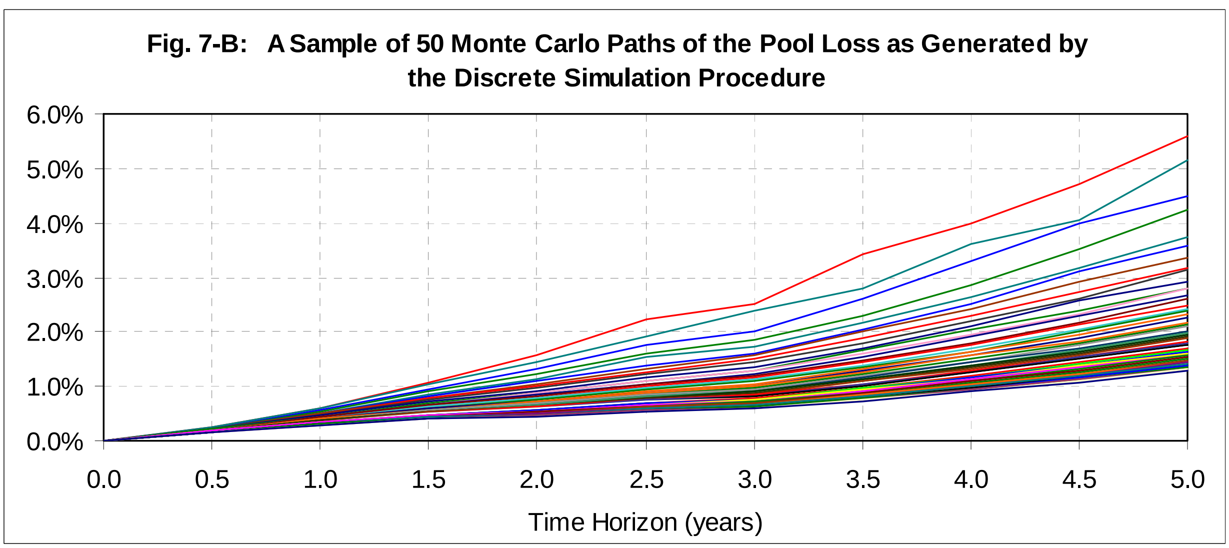

Fig. 7-B: A Sample of 50 Monte Carlo Paths of the Pool Loss as Generated by the Discrete Simulation Procedure
6.0%
5.0%
4.0%
3.0%
2.0%
1.0%
0.0%
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
4.5
5.0
Time Horizon (years)

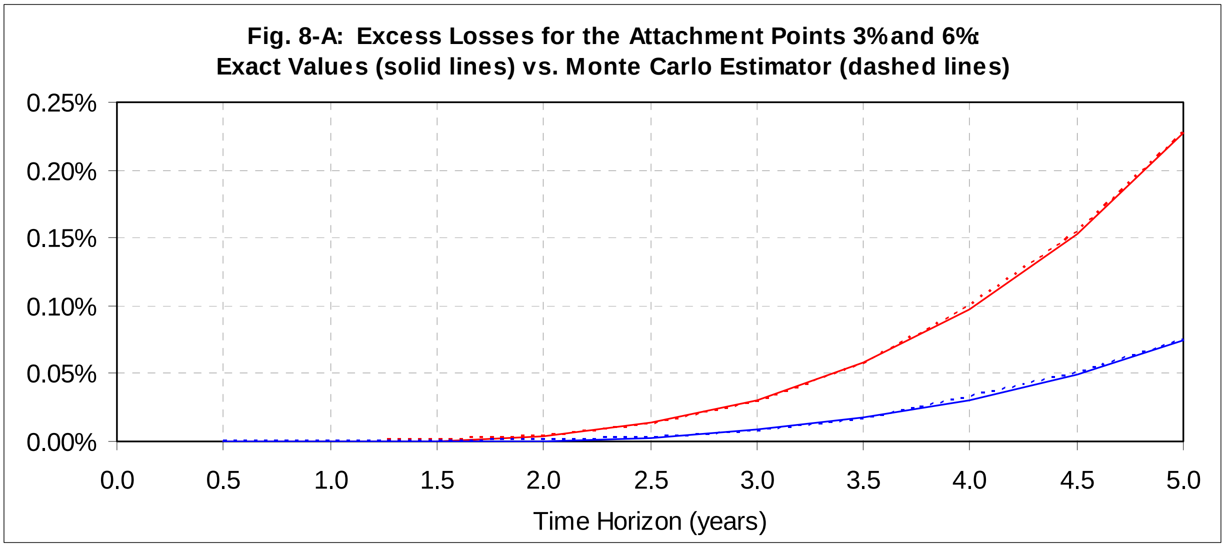
Fig. 8-A: Excess Losses for the Attachment Points 3% and 6%:
Exact Values (solid lines) vs. Monte Carlo Estimator (dashed lines)
0.25%
0.20%
0.15%
0.10%
0.05%
0.00%
0.0 0.5 1.0 1.5 2.0 2.5 3.0 3.5 4.0 4.5 5.0
Time Horizon (years)

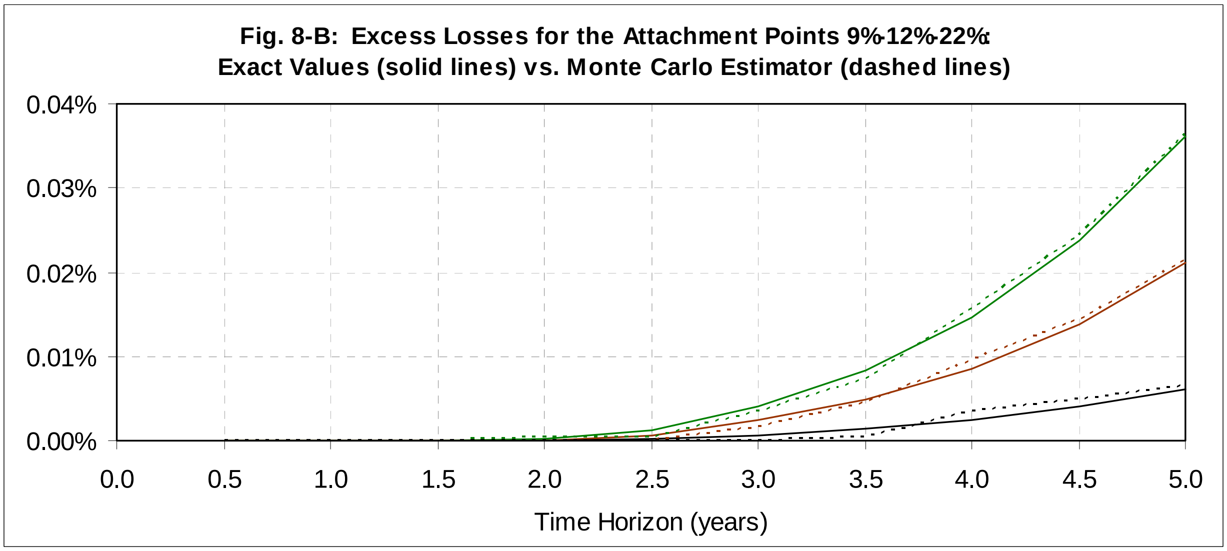
Fig. 8-B: Excess Losses for the Attachment Points 9%-12%-22%:
Exact Values (solid lines) vs. Monte Carlo Estimator (dashed lines)
0.04%
0.03%
0.02%
0.01%
0.00%
0.0 0.5 1.0 1.5 2.0 2.5 3.0 3.5 4.0 4.5 5.0
Time Horizon (years)

**Table 1: Calibration Results for the 5y CDO Tranches Spreads (iTraxx Europe) as of date Oct 17, 2005**

| Tranche | Market | Top-down' Model | LCCL Model |
|---|---|---|---|
| 3-6% | 98 | 100 | 98 |
| 6-9% | 27 | 40 | 24 |
| 9-12% | 12 | 23 | 10 |
| 12-22% | 7 | 11 | 3 |

**Table 2: Calibration Results for the 5y CDO Tranches Spreads (iTraxx Europe) as of date Apr 10, 2008**

| Tranche | Market | Basic LCCL Model | LCCL-Fat-Tail Model |
|---|---|---|---|
| 3-6% | 433 | 702 | 471 |
| 6-9% | 271 | 294 | 226 |
| 9-12% | 185 | 159 | 152 |
| 12-22% | 88 | 68 | 104 |
| 22-100% | 41 | 7 | 43 |